Every power of 2 is the sum of two prime numbers

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Abstract

In this document we break down the sequence of prime numbers into sub-sequences in order to identify some properties of these to extend to the whole sequence. In particular, we want to prove that the Goldbach Conjecture is verified for all powers of two.

1 Introduction

In a letter, in 1742, Christian Goldbach proposed to Leonhard Euler the famous conjecture, that took his name, asserting that every number greater than 5 can be written as a sum of three prime numbers. Euler answered with the actual formulation of the conjecture: every even number greater than 2 can be written as a sum of two prime numbers. Although it has been seen through computation that the conjecture is true for the first $4 * 10^{18}$ numbers[1], it remains unsolved. In this paper we show a method to prove that is true for the even numbers that are power of two. In the following sections we use a geometrical approach and some known concepts of modular arithmetic like congruences, nevertheless, we do not use that notation, also, describing some star polygons, we do not use the Schläfli symbols. The purpose of this work is to share with the community some ideas and, possibly, to collect some feedbacks.

2 Circles

Let's consider the sequences of the multiples of prime numbers where the multiples are indicated with 'x' and the non-multiples with a 'o'.
So the sequence of 2-non-multiples is:

$$\begin{array}{cccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 & \dots \\ x & o & x & o & x & o & x & \dots \end{array}$$

Sequence of 3-non-multiples:

$$x \quad o \quad o \quad x \quad o \quad o \quad x \quad o \quad o \quad \dots$$

Sequence of 5-non-multiples:

$$x \quad o \quad o \quad o \quad o \quad x \quad o \quad o \quad o \quad o \quad x \quad o \quad o \quad o \quad o \quad \dots$$

And so on...
Let's combine two sequences, the 2 and 3 non-multiples ones:

$$\begin{array}{ccccccccccccc} x & o & x & o & x & o & x & o & x & o & x & o & \dots \\ x & o & o & x & o & o & x & o & o & x & o & o & \dots \\ \hline x & o & x & x & x & o & x & o & x & x & x & o & \dots \end{array}$$

Let's call this (2,3)-non-multiple sequence for brevity $p_2\#$*-sequence*. It repeats itself every 6 elements.
Let's combine three sequences, the 2 , 3 and 5 non-multiples ones:

$$\begin{array}{cccccccccccccccc}
x & o & x & o & x & o & x & o & x & o & x & o & x & o & x & \dots \\
x & o & o & x & o & o & x & o & o & x & o & o & x & o & o & \dots \\
x & o & o & o & o & x & o & o & o & o & x & o & o & o & o & \dots \\
\hline
x & o & x & x & x & x & x & o & x & x & x & o & x & o & x & \dots
\end{array}$$

This sequence repeats itself every 30 elements. Let's call it $p_3\#$*-sequence*

Observations:

- Every sequence of this kind repeats itself after an interval which is the product of the primes $p_n\#$ that generate it.
- The sequence of the first p_n primes is the sequence of p_{n-1} combinated with the sequence of p_n.
- From the famous "Sieve of Eratosthenes" for elements less than n^2 no further prime needs to compose the sequence that are greater than $\sqrt{n}$ so, there is an interval where these sequences and the sequence of the prime numbers coincide. This is from $p_n + 1$ to $p_{n+1}^2 - 1$. However, we consider the interval between p_n^2 and $p_{n+1}^2 - 1$ in order to split the sequence of primes in n separate sub-sequences. For example, the *(2,3,5)-sequence*, from position 9 to position 24, coincides with prime numbers sequence.

Lemma 2.1 *If a number is a non-multiple in a $p_n\#$-sequence then is also a non-multiple in a $p_{n-1}\#$-sequence*

Lemma 2.2 (Sieve of Eratosthenes) *If $m > 1$ is a number not a multiple of the first prime numbers 2,3,5,...,p_n and $m < p_{n+1}^2$, m is prime.*

The number of non-multiple and non-multiple-twins for these sequences has been calculated by George Grob and Matthias Schmitt[2], that is, respectively $\prod_{i=2}^{n}(p_i - 1)$ and $\prod_{i=2}^{n}(p_i - 2)$. Since these sequences are repeated, let's close them in a circle, such that the beginning of the sequence coincide with the end of it. Let's call these objects "$p_n\#$-circles". The position of zero is on "6 o'clock" and it proceeds counterclockwise. **Figure 1** and **Figure 2** show respectively the $p_2\#$-circle and the $p_3\#$-circle.

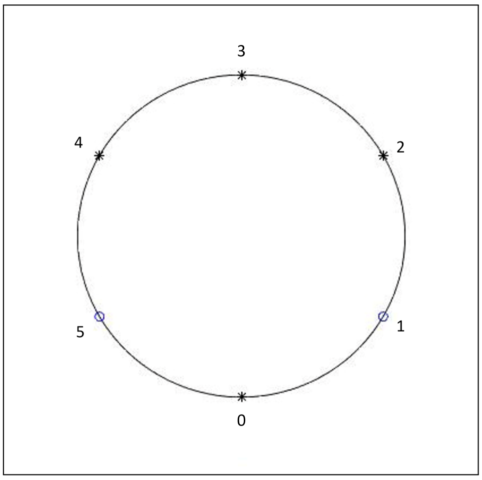

Figure 1: $p_2\#$-*circle*

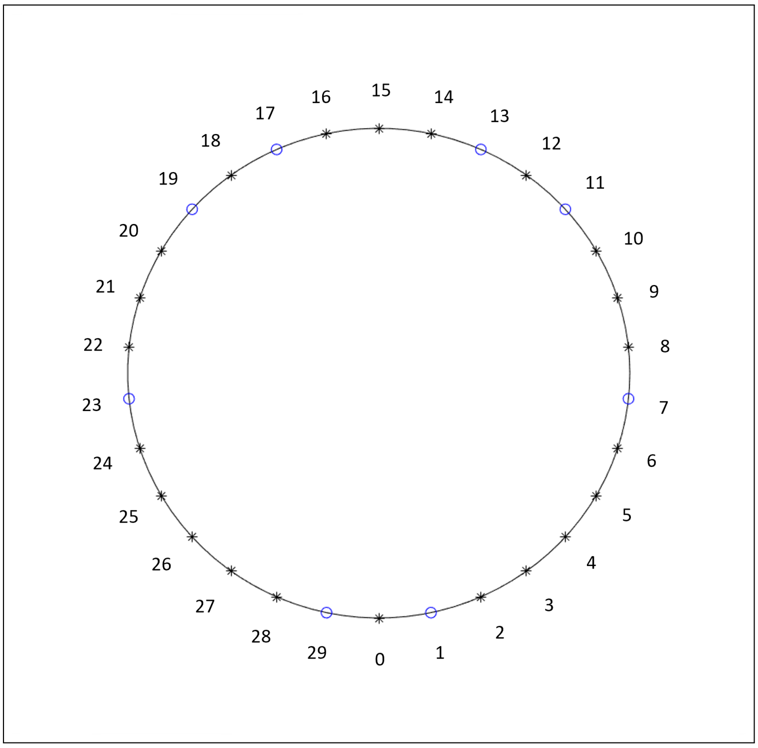

Figure 2: $p_3\#$-*circle*

Figure 3 shows also the $p_4\#$-*circle*.

Now that we have defined these objects, we can notice that they all have a vertical axis of symmetry.

Let's take the $p_3\#$-*circle* for convenience, we can numerate the dots/circles from

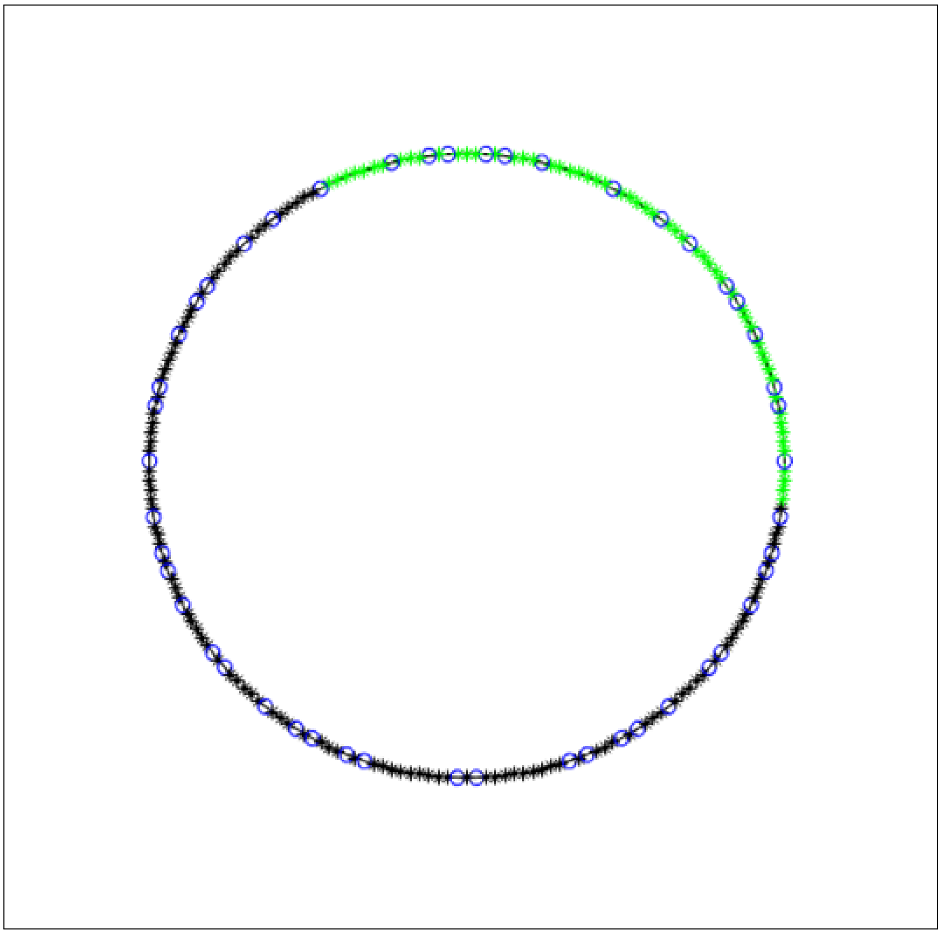

Figure 3: *$p_4\#$-circle with, highlighted in green, its separate sub-sequence of prime numbers from 49 to 120*

0 to 29.

- Starting from point 2 let's divide it in half, with a diameter.
- Do the same with the specular point, point 28.
- We call these segment "diameter generators" or only "generators".

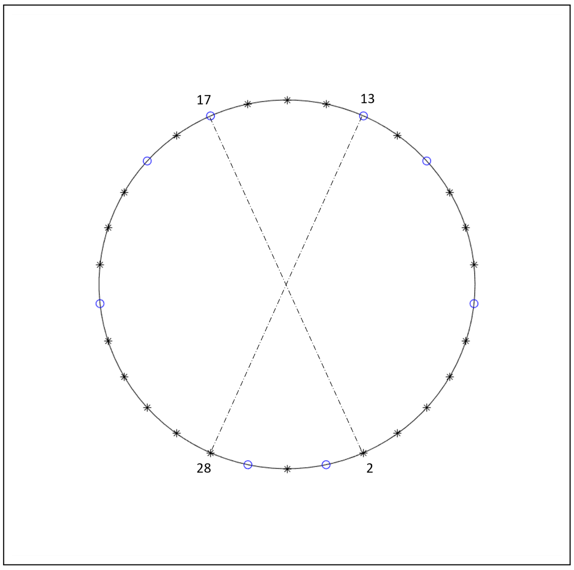

Figure 4: *$p_3\#$-circle with 2's diameter generators (one of the diameter starting from 2, the other is specular)*

Now that we have these two generators, **Figure 4**, we use them to project lines with respect to them. We use the perpendicular to these two axes to build an open broken line.

- Let's trace the perpendicular to the diameter of 2 passing through 13.
- We arrive at another odd point, this time we shoot the perpendicular to the diameter of 28.
- Let's trace another perpendicular to diameter of 2 and, as soon as we arrive at the odd on the circumference, we trace the perpendicular to the other axis.
- At the end of this process we end up on one of the points that we had intercepted with the initial diameters.

We call this result $p_3\#$*-2-connected*, **Figure 5**.

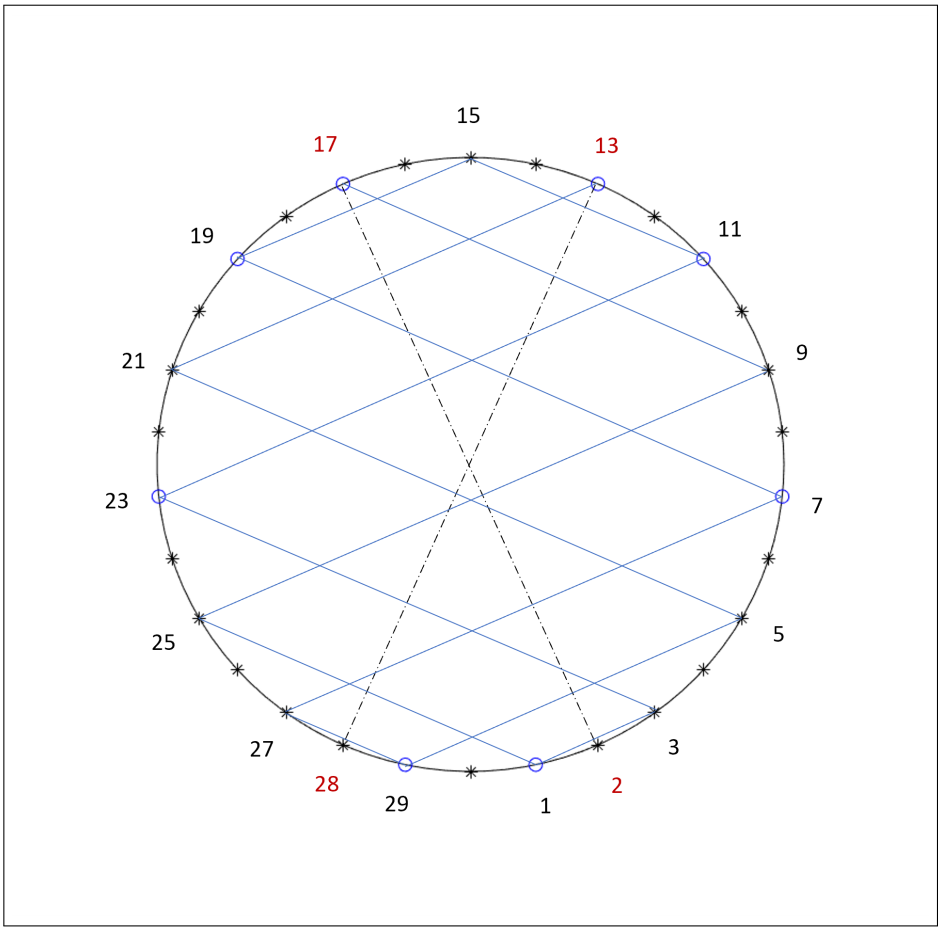

Figure 5: *p_3#-2-connected with 2's generators and perpendicular segments passing through odd numbers*

We can take every point of the circle to have two generators. Obviously some are repeated for simmetry. For example, starting from any one point among 13, 17 or 28, the resulting connection is the same as the previous one where we started from 2.
Keeping these repetitions in mind, we can still build connections for all other points of the circle (**Figure 6**).

Observation:

- We see that some broken lines intercept all odd numbers, some don't.
- The connections where we intercept all the odds with one broken line have the same sequence as the odd numbers of the p_3#-*circle*, that, starting from 1 to 29 is:

oxxoxooxooxoxxo

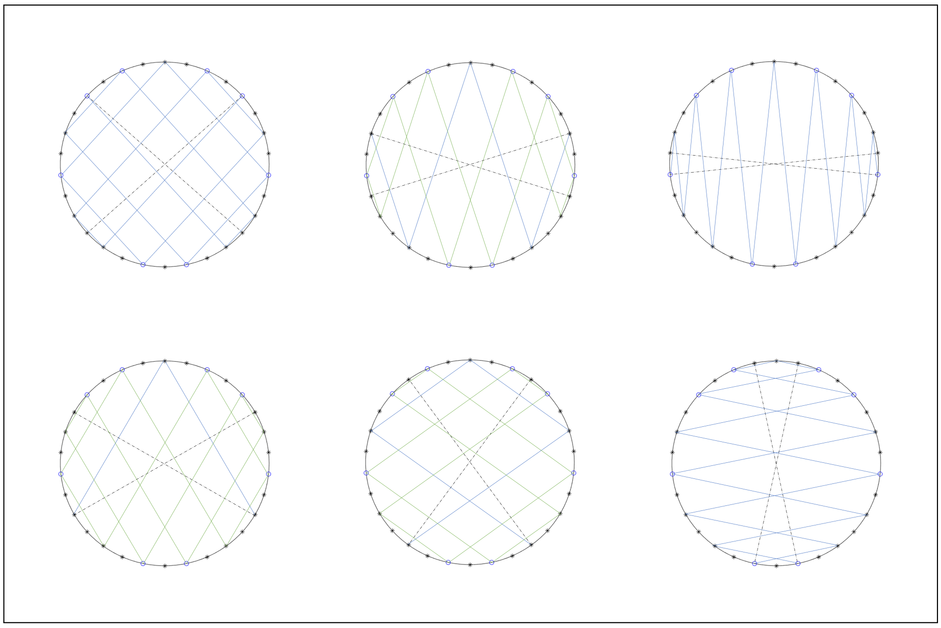

Figure 6: *$p_3\#$-$2n$-connected circles with 4,6,8,10,12 and 14 as generators respectively, the green segments are not connected with blue segments*

> We demonstrate this property in the next section. We denominate it *"original sequence"*.

We know that the numbers $3*5 \pm 2^x$ are non-multiples because $2 \nmid 15$ and $2 \mid 2^x$. Vice versa, 3 and 5 divide 15 but they don't divide 2^x.

Lemma 2.3 *Given the $p_n\#$-circle and $d, k \in \mathbb{N}$ with $d \neq 1$ and $1 \leq k \leq n$*

$$\frac{\prod_{i=1}^{n} p_i}{p_k} \pm d * p_k^x$$

is a non-multiple if and only if

$$\frac{\prod_{i=1}^{n} p_i}{p_k} \pm d$$

is a non-multiple.

Proof: If $\frac{\prod_{i=1}^{n} p_i}{p_k} \pm d$ is a non-multiple then we have that

$$d = p_k^a \; p_{n+1}^b \; p_{n+2}^c \; p_{n+3}^d \ldots \; \textit{with} \; a, b, c, d, \cdots \in \mathbb{N}$$

because the previous primes divide $\frac{\prod_{i=1}^{n} p_i}{p_k}$ and cannot divide d.
So $\frac{\prod_{i=1}^{n} p_i}{p_k} \pm d * p_k^x$ is a non-multiple too.
If $\frac{\prod_{i=1}^{n} p_i}{p_k} \pm d$ is a multiple then there exists some p_j with $1 \leq j \leq n; j \neq k$ such that $p_j \mid d$ so $\frac{\prod_{i=1}^{n} p_i}{p_k} \pm d * p_k^x$ is a multiple too. ■
The only connnections that reach all the odds are those that have as the generator axis a power of 2, because $\frac{\prod_{i=1}^{n} p_i}{2}$ (odd numbers) and 2^x are relatively prime. We will consider only this kind of circles in next sections.

3 Star polygons

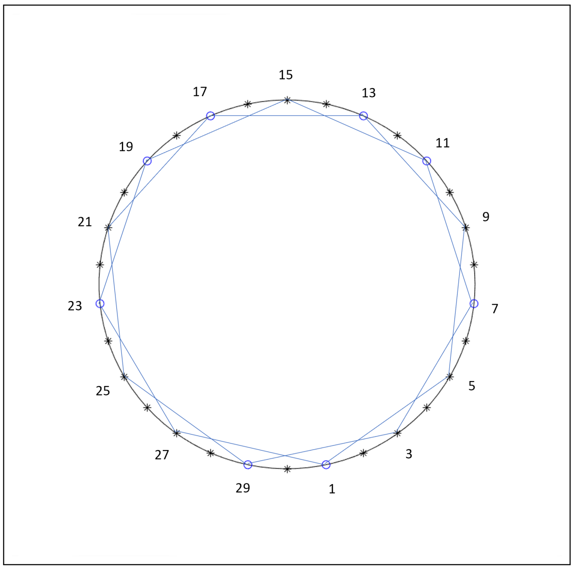

Figure 7: *p_3#-2-star*

We can take any circle of those considered. Let's take for example the p_3#-circle. Now, starting from 15, we connect all the elements moving by twice the length of the generator, in this case 2. So we connect 15 with 15-4, 15-4 with 15-8, and so on .... At the end of the process we return to the point 15. We can call the resulting figure *"p_3#-2-star"*, **Figure 7**.

We define now as "right vertex" the vertex on the right of an hypothetical observer positioned on the middle of the side of the star looking at the center of the circle, at the same time we define "left vertex" the other.

Now we can project with respect to the vertical axis of symmetry one vertex of the star to the opposite side to build a corresponding segment of the *p_3#-2-connected*, as shown in **Figure 8**.

So for each segment of a star we can obtain a segment of the connected in which one point is in common and the other is the specular so the nature ('*' or 'o') of the end-points of the segment is preserved.

If we project the left vertex of a star side we obtain a segment perpendicular to the 2-generator and if we project the right vertex we obtain a segment perpendicular to the 28-generator **Figure 9(2)**. Now we enumerate all of the sides of the star starting from that connecting 15 and 15-4, moving clockwise, and using this notation: *[first vertex, second vertex](index)*. We have [15,11](1), [11,7](2), [7,4](3), [4,1](4), [1,27](5), ... Now we project, for those sides of the star with odd index, the right vertex and, for the others, the left one. As a result, **Figure 9(4)**, we obtain the same connected figure generated in the previous section, *p_3#-2-connected*. Now we can demonstrate the following theorem.

Theorem 3.1 *The sequence of a p_n#-2^x-star and of a p_n#-2^x-connected is the same as the odd numbers in the p_n#-circle*

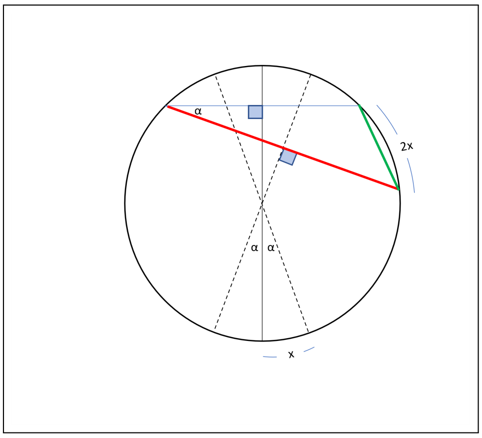

Figure 8: *The right vertex of a star side (in green) projected with respect to the vertical axis of simmetry connected with left vertex forms a 90 °angle with one generator. As a conseguence, the resulting chord (in red) is a segment of p_n#-connected*

Let's focus on the top vertex of the star, which is 15. Starting from 15 and moving by the double of the generator, by Lemma 2.3 the sequence must always be the same even if this time we move by 4 and not by 2 as in the original sequence of the circle. By the construction shown before also the connected must follow this sequence. ■

Lemma 3.2 *Given a p_n#-circle and a star polygon of generator 2^x inscribed in it, if t is the number of segments of the star polygon in which both edge points are non-multiples passing over zero then t/2 is the number of couples of non-multiples equidistant from 2^x in which one the element of the couple is between 0 and 2^x.*

Proof: we take as example the *p_3#-4-star* and we focus on the "zero position" (6 o'clock). Starting from 15 and following clockwise the sides of the star, we stop when we have zero between two consecutive non-multiples as shown in **Figure 10**. We switch the number after zero, 29, and take the specular, 1. By simmetry this must be also non-multiple. The segment connecting 7 and 1 is perpendicular to the 4-generator, so 7 and 1 are equidistant from 4 (the number 1 is not a prime obviously, we handle it in section 5). Note we can obtain the same if we project 7 instead of 29 and take the specular of the projection. So, given the side $[k, k + 2^{x+1}](i)$ it doesn't matter if i is even or odd, we can obtain a segment where both the edges are in the right half of the circle. So, one of them is between 0 and 4. Since if there exists a side of a star with non-multiples in both the edges then there exists also another that is the specular and by the previous consideration from both we obtain the same segment of equidistant non-multiple in which one the edge is between 0 and 4. We can conclude that the number of segments of this kind is half the number of the star sides connecting two non-multiples passing over zero. ■
For example, in **Figure 10** we have converted a *p_3#-4-star side* to a *p_3#-4-connected side*. In the next section we will go deep into this case where two consecutive non-multiples of a star side pass above zero. First we need some

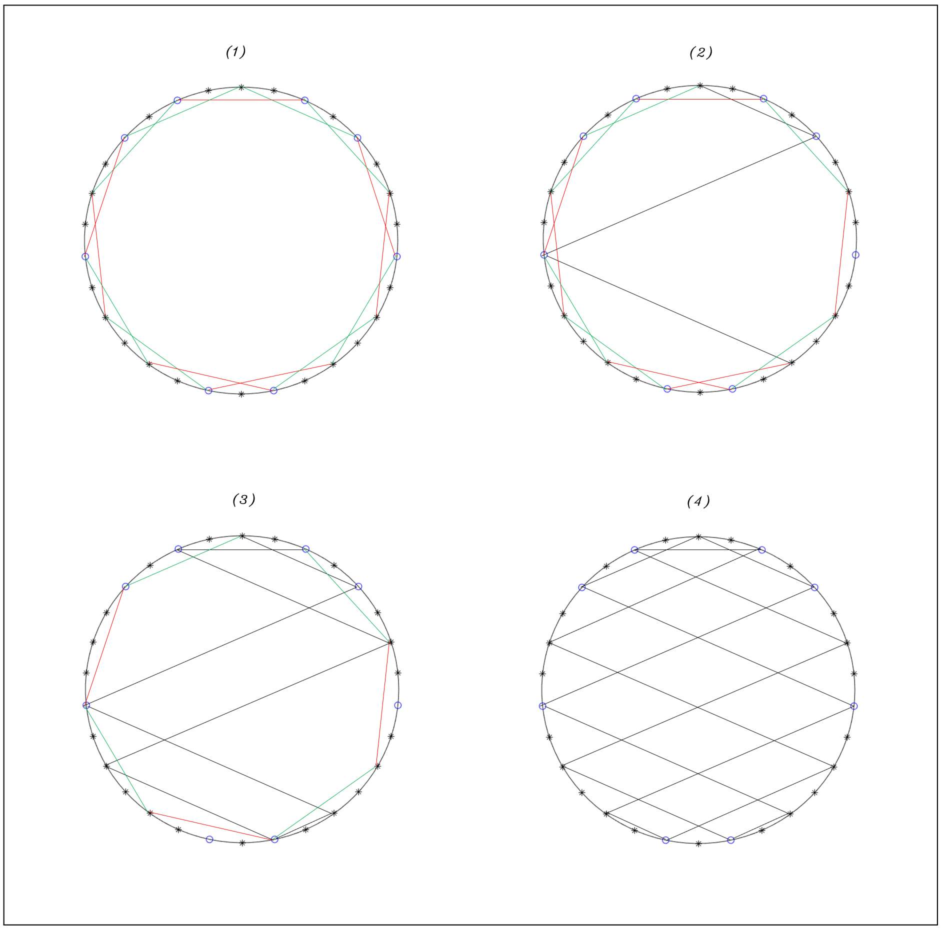

Figure 9: *Transformation of* $p_3\#$*-2-star into* $p_3\#$*-2-connected. The even sides (red) and the odd sides (green) move respectively the left and the right vertices to obtain their projection(black)*

intermediate step. We want to know how many times the sides of a star pass over zero. The length of a lap is $\sum_{i=1}^{n} p_i$, starting from a position to reach the same position moving by 2m we must cover $k * \sum_{i=1}^{n} p_i$ with minimum k such that $2m|(k * \sum_{i=1}^{n} p_i)$. If m is a power of two, k is for sure half the length of segment, that is m. So the laps we have to do are m, as shown in **Figure 11**.

Lemma 3.3 *Starting from any position in a* $p_n\#$*-circle and moving by a step of 2m, to reach the same position, if* $m \nmid (\sum_{i=2}^{n} p_i)$*, it occours to pass m times over zero.*

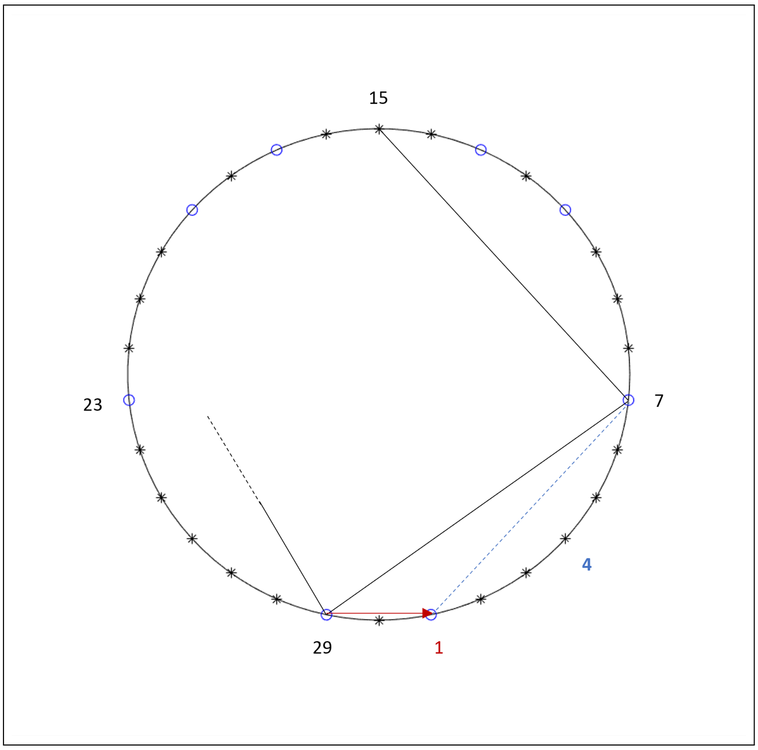

Figure 10: *From the broken line obtained in $p_3\#$-4-connected 4 is in the middle of 1 and 7, the red arrow shows the reverse transformation from $p_3\#$-4-star*

4 Polygons and twins

Theorem 4.1 *For each 2^x-gon inscribed in a $p_n\#$-circle, if 2t is the number of vertices of the 2^x-gon passing between two non-multiple-twins, there are t couples of non-multiple (a,b), such that $2^x - a = b - 2^x$.*

Considering that usually a star needs more than one lap to get closed, more than one zero position is crossed. We have as many zero positions as the laps of the star. These zeros are encountered by the star at precise points in the original sequence. We take 4 as example even this time. By Lemma 3.3 we pass over zero as many times as the number of laps we do, so they are 4. Now we do the following proportion:

$$\frac{arc\ separing\ two\ zeros}{length\ of\ 4\ laps} = \frac{arc\ separing\ two\ new\ points\ on\ circle}{length\ of\ one\ lap}$$

We have reported the 4 zeros into one single lap. If we connect them, it is like inscribing a square in the circle and the zeros are the vertices. The **Figure 12** shows this. Note that the first zero is encountered, starting from 15, after half of a lap by the star so, also here the first vertex is positioned after half of the arc after 15. The four zeros of **Figure 11** are now the four vertices of the square. Now we can extend the Lemma 3.2. In fact, if there exists some vertex of a

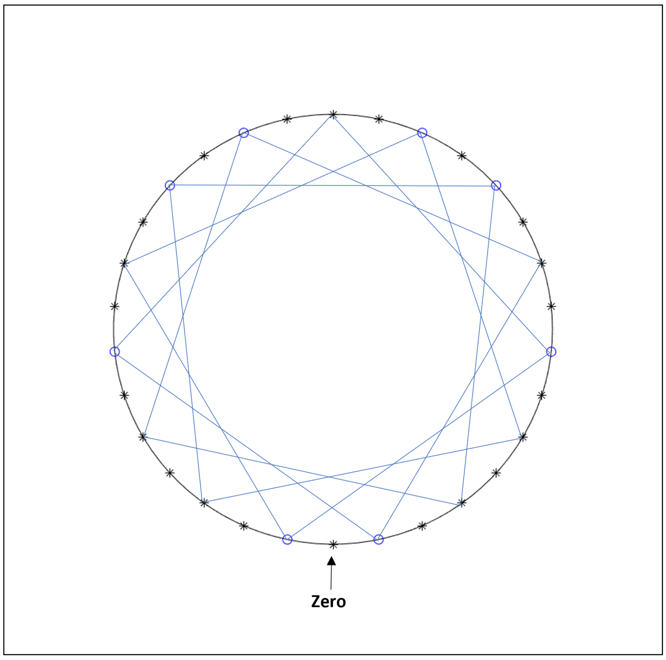

Figure 11: *$p_3\#$-4-star, the broken line pass 4 time over zero*

regular polygon with 2^x sides that lies between two twins, then we have two non-multiples such that their sum gives 2^{x+1} and the Teorem 4.1 is proved. ■

In **Figure 13** we have also the octagon. We can notice that 8 is not between two primes in this circle.

Obviously the position of the vertices, if we consider the length of the circumference as $\prod_{i=1}^{n} p_i$, do not coincides with integers because we made a proportion between not relatively prime numbers. For simplicity we call the non-multiple twins only "twins" but they are not twin primes. We want to demostrate that there always exists, for all 2^x-*gon* such that $\frac{p_{n+1}^2}{2} > 2^x > \frac{p_n^2}{2}$ at least a vertex that passes between two twins. Let's start comparing the 2^x-*gon* with the polygon of the odd multiples of p_n, which has $\frac{\prod_{i=1}^{n} p_i}{2p_n}$ sides. We can call these polygons *"n-odd-gons"*. More generically, we denominate *(n-k)-odd-gon* a polygon of $\frac{\prod_{i=1}^{n} p_i}{2p_k}$ sides. In this case the number of the sides is obtained dividing by p_n so we call this polygon *(n,n)-odd-gon*. At the same time the length of the arc whose chord is the side of the 2^x-*gon* is $\frac{\prod_{i=1}^{n} p_i}{2^x}$. For simplicity, we define as "arc" of the $2^x - gon$ that part of the circumference between two consecutive vertex. Now we want to know where the vertices of the *(n,n)-odd-gon* intersect these arcs.

If we report into a single segment the points where the vertices of the *odd-gon* intersect the arcs of the 2^x-*gon* we have points at equal intervals, **Figure 14**. So we call $A = [0, \frac{\prod_{i=1}^{n} p_i}{2^x}]$ the arc domain where we project the odd-gon vertices X_i of the circle domain C. Let's call these intervals a.

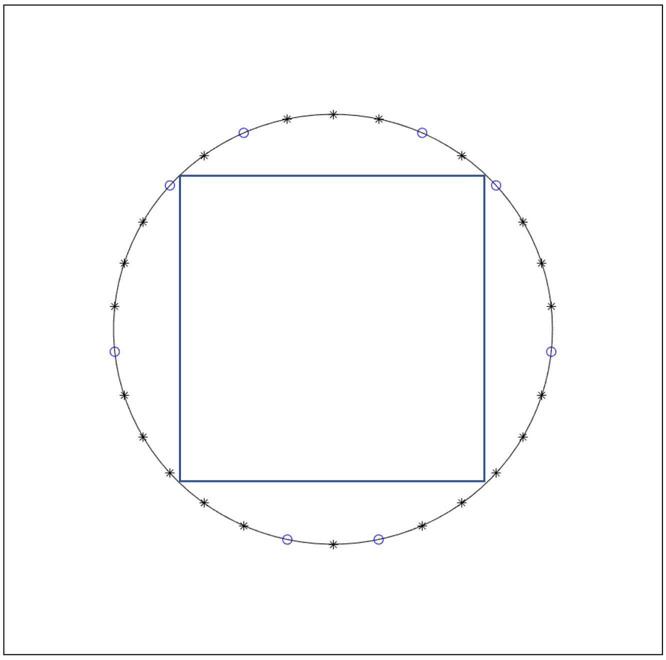

Figure 12: *Polygon of* 2^2 *sides (square) which vertices pass through 2 couples of twins. Note that 4 have exactly the half of couples, one in this case, of equidistant non-multiples (the cirlcles 'o')*

A= V<–$\frac{a}{2}$–|—-a—-|–. . . –|—-a—-|—-a—-|–. . . –|—-a—-|–$\frac{a}{2}$–>V

V=2^x-*gon* vertex

the length of a is the side of the 2^x-*gon* divided by the number of the vertices of the *(n,n)-odd-gon*. In general we have that a is:

$$a_n = \frac{\prod_{i=1}^{n} p_i}{2^x \frac{\prod_{i=1}^{n} p_i}{2p_n}} = \frac{2p_n}{2^x}$$

For each $X_i \in C$ there exists an $f(X_i) = (\frac{1}{2} + k) * a = Y_k \in A$ and k and i are not necessary equal to each other so the sequence of the elements on the arc is not the same as that of the vertices of the odd-gon in the circle.

If a vertex of 2^x-*gon* lies between two twins, both are at a distance at most two from the vertex. Now we define a subset $B \in A$ where $B = [0, 2] \cup [\frac{\prod_{i=1}^{n} p_i}{2^x} - 2, \frac{\prod_{i=1}^{n} p_i}{2^x}]$. We want to know how many Y_k are in B because they are as many as the X_i which are distant less than 2 from a vertex of the 2^x-gon in C. To do this we can append the first half of A $[0, \frac{\prod_{i=1}^{n} p_i}{2*2^x}]$ to the second half of A $[\frac{\prod_{i=1}^{n} p_i}{2*2^x}, \frac{\prod_{i=1}^{n} p_i}{2^x}]$ as figure below, we call this A'. As consequence we define B' as a continuous subset of A'.

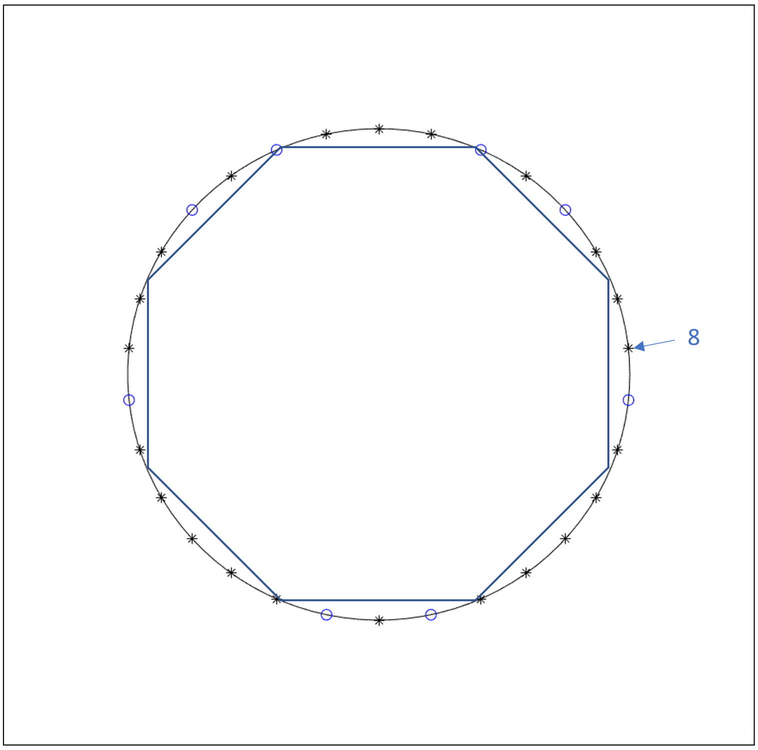

Figure 13: *Vertices of polygon of 2^3 sides (octagon) don't pass through any couple of twins. Note that 8 has no equidistant non-multiples*

|—-a—-|–. . . –|—-a—-|–$\frac{a}{2}$–>V<–$\frac{a}{2}$–|—-a—-|–. . . –|—-a—-| A'

<———2———>V<———2———> B'

B' is the union of two subset of range 2. So the range of it is 4.
The number of intersections of the *(n-n)-odd-gon* in this range B' are *b*:

$$b_n = \begin{cases} floor(4/a_n) & if\ even\ number \\ floor(4/a_n)+1 & otherwise \end{cases} \tag{1}$$

So b is always even. Since $\frac{4}{a_n}$ is not an even integer we have to add a $\delta \in (-1, 1)$:

$$b_n = \frac{4}{a_n} + \delta_n$$

$\boxed{\textbf{Open point: } \delta \text{ is not known.}}$

Getting back from A to C, b_n is the number of vertices of the *(n,n)-odd-gon* that are distant less than two from a vertex of the 2^x*-gon*. We can note that, if a vertex X_i of the (n,n)-odd-gon is distant less than two from a vertex V of the 2^x*-gon*, then X_i is one of the two nearest odd numbers with respect to V, and by construction it's divisible by p_n: it follows that V is not between a pair of twin non-multiples. Our goal will be to calculate how many twin non-multiples

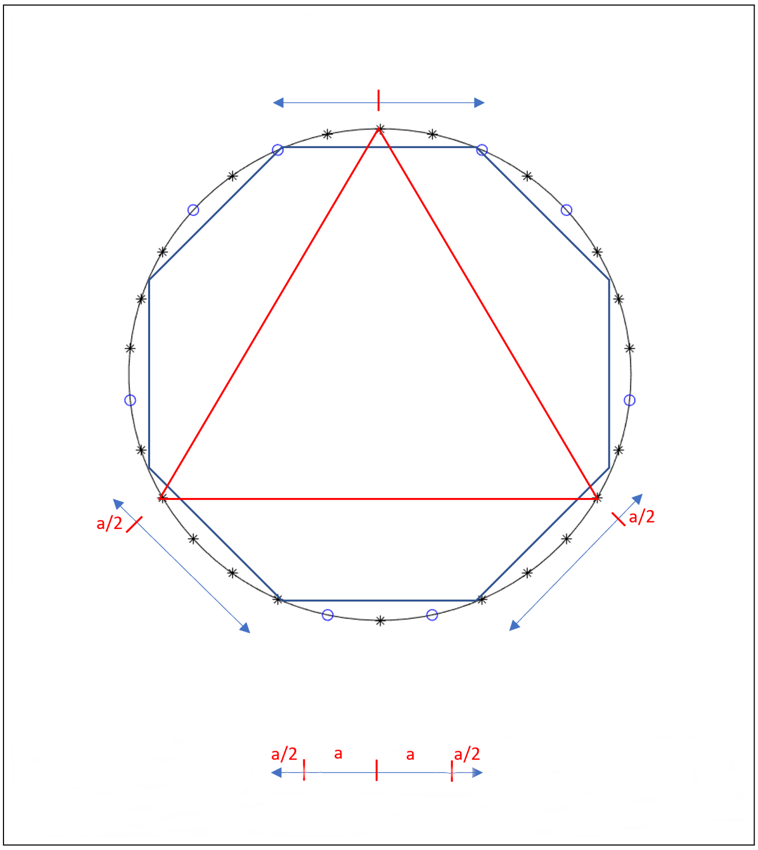

Figure 14: *Vertices of (3,3)-odd-gon divide the arc of 2^x-gon into equal parts (two halves of part on the edges)*

are eliminated this way by each of $p_2, p_3, \ldots, p_n$, in order to prove that at the end of the process some twin couples remain.
Of course, there are some overlappings. For example, $p_n p_k$ is a vertex of the *(n,n)-odd-gon*, but it's also a vertex of the *(n,k)-odd-gon*. Up to now we have considered the polygon drawn by p_n for odd numbers (*(n,n)-odd-gon*) therefore also including the intersections already made by the previous prime numbers. To avoid counting twice these common points and to get the effective number of twin couples covered by the entry of p_n, the portion $1/p_k$ must be removed for every $1 < k < n$ from the vertices of the *(n,n)-odd-gon*, in other words we remove the common vertices between *(n,n)-odd-gon* and *(n-k)-odd-gon*. The number of these common vertices is the $gcd = \frac{\prod_{i=1}^{n} p_i}{2 p_n p_k}$ between the vertices of the two odd-gons and they form an other odd-gon of $p_{n-1}\#$-circle.
Example: $\frac{2*3*5*7*11}{2*11}$ and $\frac{2*3*5*7*11}{2*7}$ give $3*5 = \frac{2*3*5*7}{2*7} = \frac{2*3*5*7*11}{2*11*7}$.
We call this *main (n-1,k)-odd-gon*.
The number of vertices to remove is $c_{n,k} = \frac{4}{a_n p_k} + \delta_{n,k}$. Also here we have a non-integer result so we added a δ to force the result to be an even number. This result is also $c_{n,k} = \frac{b_n - \delta_n}{p_k} + \delta_{n,k}$.

In addition to this, the portion of those couples of twins in which one of

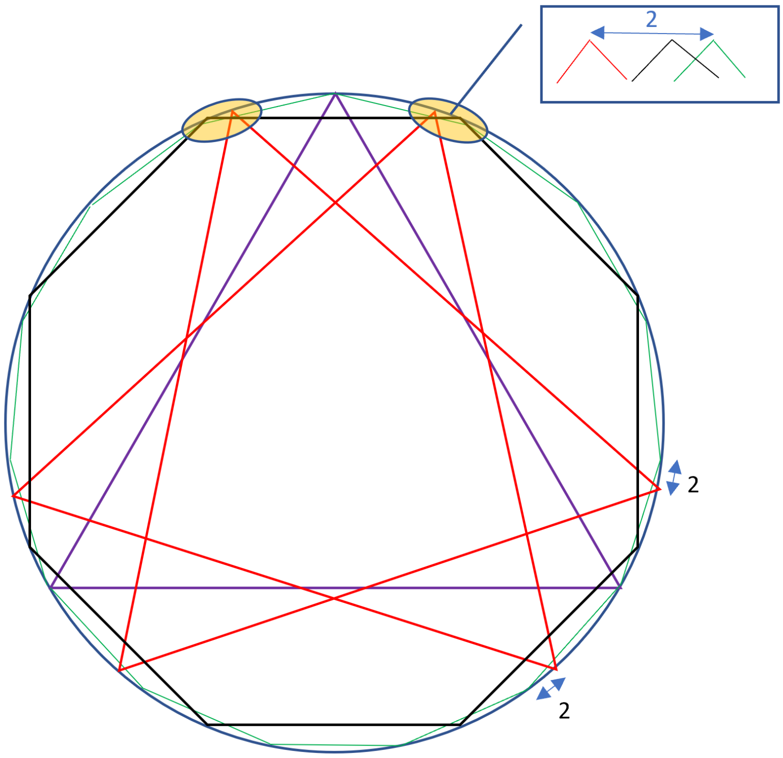

Figure 15: *Representation of a main-odd-gon(purple) and left and right odd-gons (red) generated from two odd-gons (one of which is not represented and the other is the one in green). The main-odd gon is the connection of the common vertices between the two odd-gons. The left and the right odd-gons have all the vertices in common with the second odd-gon (not represented) and, at the same time, their vertices have a distance of two from the first odd-gon(green). The couples of twins where the vertices of 2^x-gon (black) lie in a distance of two from main, left or right odd-gon are not subtracted twice in the count of the total remaining couples.*

the element is covered by *(n,n)-odd-gon* and the other element has already been covered by the other *(n,k)-odd-gons* must also be removed. To do this, we take the vertices of *(n,k)-odd-gon* that are 2 to the **left** from the *(n,n)-odd-gon*. Among these, however, are kept only those vertices staying on the **right** within a range of 2 of the 2^x*-gon*. Summarizing:

- we take a $p_n\#$*-circle* and the *(n,n)-odd-gon*
- we take the *(n,n-1)-odd-gon*
- we take all the elements of the *(n,n-1)-odd-gon* that are in a range of 2 from the left of a vertex of the *(n-n)-odd-gon*

- the connection of these elements is another odd-gon, we call this *left (n-1,n-1)-odd-gon*
- we take the elements of this resulting odd-gon that are in a range of 2 from the right of the 2^x*-gon*
- we finally find the vertices of the 2^x*-gon* that have on the right a *(n,n)-odd-gon* vertex and on the left a *(n,n-1)-odd-gon* which are distant 2 each other.

Now we do the following proportion:

$$\frac{2}{length\ of\ the\ arc\ of\ the\ 2^x gon} = \frac{d}{\substack{(n,n\text{-}1)\text{-}odd\text{-}gon\ vertices\ that\ are\ within\ 2\\ from\ the\ left\ of\ the\ (n,n)\text{-}odd\text{-}gon)\ vertices}}$$

This number of pairs d must to be an integer:

$$\frac{2}{\frac{\frac{\prod_{i=1}^n p_i}{2^x}}{\frac{\prod_{i=1}^n p_i}{2p_n p_{n-1}}}} < d_{n,n-1} < \frac{2}{\frac{\frac{\prod_{i=1}^n p_i}{2^x}}{\frac{\prod_{i=1}^n p_i}{2p_n p_{n-1}}}} + 1$$

Where $\frac{\prod_{i=1}^n p_i}{2^x}$ is the length of the arc of the $2^x - gon$ and $\frac{\prod_{i=1}^n p_i}{2p_n p_{n-1}}$ is the number of vertices of the *(n,n-1)-odd-gon* that are 2 from the left of the *(n,n)-odd-gon*. Note that this is the same number of those vertices in common between the two odd-gons.

$$2\frac{\frac{1}{2p_n p_{n-1}}}{\frac{1}{2^x}} < d_{n,n-1} < 2\frac{\frac{1}{2p_n p_{n-1}}}{\frac{1}{2^x}} + 1$$

$$\frac{2^x}{p_n p_{n-1}} < d_{n,n-1} < \frac{2^x}{p_n p_{n-1}} + 1$$

By simmetry we have to double this result, considering the specular case of left range, and we obtain the *right (n-1,n-1)-odd-gon*. We consider also the closest even number so:

$$2d_{n,n-1} = 2\frac{2^x}{p_n p_{n-1}} + \delta_{n,n-1}$$

Now we can join all the partial results and calculate t:

$$t_{n,n-1} = b_n - c_{n,n-1} - 2d_{n,n-1} = 2^x\frac{2}{p_n} - 2^x\frac{2}{p_n p_{n-1}} - 2\frac{2^x}{p_n p_{n-1}} + \delta_n - 2\delta_{n,n-1} =$$

$$= 2^x\frac{2}{p_n}\Big(1 - \frac{1}{p_{n-1}} - \frac{1}{p_{n-1}}\Big) + \delta_n - 2\delta_{n,n-1} = 2^x\frac{2}{p_n}\Big(1 - \frac{2}{p_{n-1}}\Big) + \delta_n - 2\delta_{n,n-1}$$

Once we have obtained three odd-gons, from *(n,n)-odd-gon* and *(n,n-1)-odd-gon* as shown in the **Figure 15**, we can repeat the same operations for the each of the resulting odd-gons and the *(n,n-2)-odd-gon*.

$$t_{n,n-1,n-2} = 2^x\frac{2}{p_n} - 2^x\frac{2}{p_n p_{n-1}} - 2\frac{2^x}{p_n p_{n-1}} - 2^x\frac{2}{p_n p_{n-2}} - 2\frac{2^x}{p_n p_{n-2}} + 2^x\frac{2}{p_n p_{n-1} p_{n-2}} +$$

$$+2\frac{2^x}{p_np_{n-1}p_{n-2}}+2\frac{2^x}{p_np_{n-1}p_{n-2}}+2\frac{2^x}{p_np_{n-1}p_{n-2}}+\delta_n-2\delta_{n,n-1}-2\delta_{n,n-2}+4\delta_{n,n-1,n-2}=$$

$$=2\frac{2^x}{p_n}\Big(1-\frac{2}{p_{n-1}}\Big)\Big(1-\frac{2}{p_{n-2}}\Big)+\delta_n-2\delta_{n,n-1}-2\delta_{n,n-2}+4\delta_{n,n-1,n-2}$$

The four addends $2^x\frac{2}{p_np_{n-1}p_{n-2}}+\delta_{n,n-1,n-2}$ count the common vertices between:

- main (n,n-1)-odd-gon and main (n,n-2)-odd-gon
- main (n,n-1)-odd-gon and left and right (n,n-2)-odd-gons
- main (n,n-2)-odd-gon and left and right (n,n-1)-odd-gons
- left (n,n-1)-odd-gon and left (n,n-2)-odd-gon; right (n,n-1)-odd-gon and right (n,n-2)-odd-gon

Note that left (n,n-1)-odd-gon and right (n,n-2)-odd-gon cannot have common couple of twins because they are -2 and +2 with respect to the vertices of (n,n)-odd-gon. Considering the combination for each $1 < k < n$:

$$t_n = 2^x\frac{2}{p_n}\Big(1-\frac{2}{3}\Big)\Big(1-\frac{2}{5}\Big)\Big(1-\frac{2}{7}\Big)\Big(1-\frac{2}{11}\Big)\cdots+\delta_{n,tot}$$

t_n is the upper bound of the number of couples lost by a 2^x*-gon* when passing from
the p_{n-1}#*-circle* to the p_n#*-circle*. Unfortunately, until now, we are not able to define a non-exponential upper bound for $\delta_{n,tot}$. In the next section we assume an hypothetical upper bound for the summation of all these δ, but this is not proved yet.

5 Statement and hypothesis

Theorem 5.1 $\forall n, x \in \mathbb{N} : \frac{p_{n+1}^2}{2} > 2^x > \frac{p_n^2}{2}$ *$\exists p, q$ prime numbers with $q > p$:* $q < p_{n+1}^2 \wedge \frac{p+q}{2} = 2^x$

Starting from theorem 4.1, having also calculated t in previous section, we have that 2^x is the number of vertices between twins of a 2^x*-gon* at the beginning assuming that there are no primes, except 2. Now we have to subtract every combination of *odd-gons* for every *circle* $p_2\#$,$p_3\#$,$p_4\#$,...,$p_n\#$ and prove that thenumber of remaining couples is a positive number. We can also consider to check the property only for each x such that $2^x > \frac{p_n^2}{2}$ because, if it's true for such x, it is also true for $\frac{p_n^2}{2} > 2^x > \frac{p_{n+1}^2}{2}$. The number of remaining vertices for the n-th circle is

$$r_n = 2^x - \sum_{i=2}^{n} t_i = 2^x - \sum_{i=2}^{n} \left(2^x \Big(\frac{2}{p_i} \prod_{j=2}^{i-1} (\frac{p_j - 2}{p_j}) \Big) + \delta_{i,tot} \right)$$

$$r_n = 2^x \left(1 - \sum_{i=2}^{n} \Big(\frac{2}{p_i} \prod_{j=2}^{i-1} (\frac{p_j - 2}{p_j}) \Big) \right) - \delta$$

where $\delta = \sum_{i=2}^{n} \delta_{i,tot}$.
r must be greater or equal than 4 (we have to divide by 2 this result to obtain the couples of equidistant non-multiples, also we can't admit the couple with 1 because is not a prime) for every $2^x > \frac{p_n^2}{2}$.

So if r is greater or equal to 4 there will be at least one couple of prime numbers equidistant from 2^x.

$$r_n = 2^x \left(1 - \sum_{i=2}^{n} \Big(\frac{2}{p_i} \prod_{j=2}^{i-1} (\frac{p_j - 2}{p_j}) \Big) \right) - \delta > \frac{p_n^2}{2} \left(1 - \sum_{i=2}^{n} \Big(\frac{2}{p_i} \prod_{j=2}^{i-1} (\frac{p_j - 2}{p_j}) \Big) \right) - \delta \geq 4$$

We can simplify the summation s_n.

$$s_n = \sum_{i=2}^{n} \Big(\frac{2}{p_i} \prod_{j=2}^{i-1} (\frac{p_j - 2}{p_j}) \Big) = 1 - \prod_{i=2}^{n} \frac{p_i - 2}{p_i}$$

Proof:
Since we have that

$$s_2 = \sum_{i=2}^{2} \Big(\frac{2}{p_i} \prod_{j=2}^{i-1} (\frac{p_j - 2}{p_j}) \Big) = 1 - \prod_{i=2}^{2} \frac{p_i - 2}{p_i} = \frac{2}{3}$$

And also

$$s_{n+1} = 1 - \prod_{i=2}^{n} \frac{p_i - 2}{p_i} + \frac{2}{p_{n+1}} \prod_{j=2}^{n} \frac{p_j - 2}{p_j} = 1 - (1 - \frac{2}{p_{n+1}}) \prod_{i=2}^{n} \frac{p_i - 2}{p_i} = 1 - \prod_{i=2}^{n+1} \frac{p_i - 2}{p_i}$$

then the equality is true. Replacing in the previous inequality:

$$\frac{p_n^2}{2}\left(1-\left(1-\prod_{i=2}^{n}\frac{p_i-2}{p_i}\right)\right)-\delta \geq 4$$

$$\frac{p_n^2}{2}\left(\prod_{i=2}^{n}\frac{p_i-2}{p_i}\right) \geq 4+\delta$$

Note that $\prod_{i=2}^{n}\frac{p_i-2}{p_i}$ is the ratio between the number of twins in the circle and the length of the circle.
An estimation of it was made by Eric Naslund[3]. However, we don't explode the term here.

$$p_n\left(\prod_{i=2}^{n}\frac{p_i}{p_{i-1}}\right)\left(\prod_{i=2}^{n}\frac{p_i-2}{p_i}\right) \geq 4+\delta$$

Now we formulate the following hypothesis

Conjecture 5.2 $\delta < p_n - 4$

In **Table 1** we have some values of δ. Assuming this conjecture true Theorem 5.1 is true because the inequality:

$$\prod_{i=2}^{n}\frac{p_i}{p_{i-1}} > \prod_{i=2}^{n}\frac{p_i}{p_i-2}$$

holds since:

$$\frac{p_i}{p_{i-1}} \geq \frac{p_i}{p_i-2}$$

for each i and the equality holds if p_i and p_{i-1} are twin primes. If $r_n \geq 4$ then the 2^x-*gon* has at least four vertices that lie between twins in the $p_n\#$-circle and by Theorem 4.1 there exist at least two couples of non-multiples equidistant from 2^x, for every power of two $\frac{p_n^2}{2} < 2^x < \frac{p_{n+1}^2}{2}$. If we discard the possible couple of 1, the elements of the remaining couple are prime by Lemma 2.1 and Lemma 2.2 then the Theorem 5.1 is proved.

x	n	$2^x\prod_{i=2}^{n}\frac{p_i-2}{2}$	w(x,n)	δ
3	2	2.66666	2	+0.66666
4	3	3.2	4	-0.8
5	4	4.57143	6	-1.42857
6	5	7.48052	8	-0.51948
7	6	12.65934	14	-1.34066
8	8	19.98843	18	+1.98843
9	11	31.79086	38	-6.20914

Table 1: *Values of δ for x such that $\frac{p_n^2}{2} < 2^x < \frac{p_{n+1}^2}{2}$. w(x,n) is the exact number of couples of vertices of the 2^x-gons between two twins in the $p_n\#$-circle.*

References

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