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***Investigation of Hardy-Littlewood and of Goldbach conjectures with the primality theorems of Congruence and of Complementary Congruence***

***Investigation of the Hardy-Littlewood and Goldbach conjectures with the Congruence Primality Theorems and Complementary Congruence***

***Abstract***

***This article provides novel insights on Hardy-Littlewood's conjecture (infinity and distribution of twin primes) and on Goldbach's conjecture; this work is primarily based on two primality theorems of congruence and of compcongruence. Study results in demonstration of Hardy-Littlewood and the Goldbach conjectures.The approach taken also opens up new areas of possible research in the field of Number Theory.***

***A study of the Hardy-Littlewood conjecture (infinity and distribution of first twins) and the Goldbach conjecture is developed in the article; it is based primarily on two primality theorems of congruence and compcongruence. The study arrives at the proof of the Hardy-Littlewood and Goldbach conjectures. In addition to the results achieved, the study opens up new areas of possible research in the field of Number Theory.***

## 1 Congruence Primalities

### 1.1 The Congruence of Natural Numbers

As is well known, the congruence relation [1.2.1 of (a)] modulus m is an equivalence relation defined on the set of integers Z as follows: if m is a fixed integer greater than 1, two integers a and b are said to be congruent modulus m if m|(a - b); m is called the modulus of congruence and is denoted by a ≡ b (mod m).
In the field of natural numbers, it can also be equivalently stated that a ≡ b (mod m) if a and b give the same remainder in the integer division by m.
For example, 24 ≡ 10 (mod 7) because they both give remainder 3 in the integer division by 7. All numbers congruent with each other modulo m constitute an equivalence class, called the congruence class modulo m: two natural numbers belong to the same congruence class if and only if they are congruent modulo m, that is, if and only if they divide by m and give the same remainder r. If, as in the example, the modulus is 7, seven classes are thus formed (as many as there are possible remainders in the division by 7) as follows [0], [1], [2], [3], [4], [5], [6]. Always limiting ourselves to the subset of Z consisting of the natural numbers, to establish to which class modulo m one of them belongs we divide it by m, the remainder indicating the class.
It should be emphasised that for each m it is always the case that $[m]_{\text{mod m}} = [0]_{\text{mod m}}$ .

**Remark 1.1.2** From Number Theory we know that any natural number n will only be non-prime if it is divisible by one or more prime numbers less than or equal to the $\sqrt{n}$. Since all even natural numbers, except 2, are non-prime because they are divisible by 2, it can be asserted that any odd natural number n > 4 will only be non-prime if it is divisible by one or more prime numbers odd less than or equal to . $\sqrt{n}$.

From here on, the variables p, $p_1$, $p_2$, . . . $p_i$ always denote prime numbers and $\mathbb{P}(M)$ the set of odd prime numbers less than or equal to the number M.

### 1.2 Congruence Primality Theorem

**Enunciation 1.2.1** *$\forall N_0, n_0 \in N$ with $N_0 \geq 3$, $0 \leq n_0 \leq N_0 - 3$ and even if $N_0$ is odd or vice versa, with $\mathbb{P}(\sqrt{(N_0 - n_0)})$ set of odd prime numbers $\leq \sqrt{(N_0 - n_0)}$, a necessary and sufficient condition for $N_0 - n_0$ to be a prime number is that $n_0 \not\equiv N_0 \pmod{p_i}$ $\forall p_i \in \mathbb{P}(\sqrt{(N_0 - n_0)})$ or that $\mathbb{P}(\sqrt{(N_0 - n_0)})$ is an empty set.*

**Proof.** According to the congruence of natural numbers (1.1) if $N_0$ and $n_0$ do not belong to the same congruence class modulo $p_i$ for all $p_i \in \mathbb{P}(\sqrt{(N_0 - n_0)})$, this means that $N_0 - n_0$ (an always odd natural number) is not divisible by any odd prime number less than or equal to the $\sqrt{(N_0 - n_0)}$ and that therefore, according to observation (1.1.2), $N_0 - n_0$ is a prime number. If instead $\mathbb{P}(\sqrt{(N_0 - n_0)})$ results in an empty set (with $n_0 = N_0 - 3$, $N_0 - 4$, $N_0 - 5$, $N_0 - 6$, $N_0 - 7$, $N_0 - 8$) the number $N_0 - n_0$ cannot be divided by any prime and is therefore prime.
Conversely, if $N_0 - n_0$ is a prime number, it will not be divisible by any other lower, equal or non-existent odd prime number of the $\sqrt{(N_0 - n_0)}$ and therefore $N_0$ and $n_0$ will always result non congrui $\forall p_i \in \mathbb{P}(\sqrt{(N_0 - n_0)})$.

We set $n_0 \leq N_0 - 3$ because with $n_0 = N_0 - 1$ one would have that $N_0 - n_0 = 1$ which, as is known, is neither a prime nor a compound number, and with $n_0 = N_0 - 2$ one would have that $N_0$ and $n_0$ would both be even or odd contrary to the hypothesis. In order then to prevent $n_0$ from taking negative values, it must be $N_0 \geq 3$.

**Remark 1.2.2** *If, instead of referring to the set $\mathbb{P}(\sqrt{(\boldsymbol{N_0 - n_0})})$ we want to refer, for the needs of successive demonstrations, to the set $\mathbb{P}(\sqrt{\boldsymbol{N_0}})$, the theorem (1.2.1) is transformed into the corollary (1.2.3)*

Given a number $N_0 \in N$, a number $n_0 \in N$, smaller than $N_0$ and such that $(N - n_{00})$ is odd is called the **Prisotto of $N_0$** if it turns out that $n_0 \not\equiv N_0 \pmod{p_i}$ $\forall p_i \in \mathbb{P}(\sqrt{(N_0)})$.

**Corollary 1.2.3** *$\forall N_0, n_0 \in N$ with $N_0 \geq 9$, $0 \leq n_0 \leq N_0 - p_{max}$ and even if $N_0$ is odd or vice versa, with $\mathbb{P}(\sqrt{(N_0)})$ set of odd prime numbers $\leq \sqrt{(N_0)}$ and with $p_{max}$ prime number higher than $\mathbb{P}(\sqrt{(N_0)})$, a necessary and sufficient condition for $N_0 - n_0$ to be a prime number is that $n_0$ is a number prisotto of $N_0$.*

**Proof.** substituting $\mathbb{P}(\sqrt{(N_0)})$ a $\mathbb{P}(\sqrt{(N_0 - n_0)})$, in contrast to theorem (1.2.1), the numbers $n_0$ smaller than $N_0$ and belonging to the interval $[N_0 - p_{max} ., N_0 - 3]$ are not considered since they all have at least one congruence class mod $p_j$, with $p_j \in \mathbb{P}(\sqrt{(N_0)})$, equal to that of the same modulus of $N_0$. In fact for the $n_0 \in [N_0 - p_{max} ., N_0 - 3]$, $N_0 - n_0$ will belong to the interval $[3, p_{max}]$ and thus be equal to a prime or compound number belonging to this interval; in the first case according to modular arithmetic if $N_0 - n_0 = p_j$, with $p_j \in \mathbb{P}(\sqrt{(N_0)}) \subset [3, p_{max} .]$ this implies that $[N_0] \bmod p_j - [n_0] \bmod p_j = [p_j] \bmod p_j = [0]$ whence the congruence mod $p_j$ of $n_0$ with $N_0$; if instead $N_0 - n_0$ is equal to a compound number $m * p_j$, with $p_j \in \mathbb{P}(\sqrt{(N_0)}) \subset [3, p_{max} .]$, we will have that $[N_0] \bmod p_j - [n_0] \bmod p_j = [m] \bmod p_j * [p_j] \bmod p_j = [m] \bmod p_j * [0] = [0]$ whence the congruence mod $p_j$ of $n_0$ with $N_0$.
Conversely, if $N_0 - n_0$ is a prime number, belonging to the interval $]p_{max}, N_0]$, it as prime will not be divisible by any other odd prime number less than or equal to $p_{max}$ and thus the $\sqrt{(N_0)}$ and therefore $N_0$ and $n_0$ will always be non congrui $\forall p_i \in \mathbb{P}(\sqrt{(N_0)})$.

He placed himself $N_0 \geq 9$ in quanto per valori inferiori $p_{max}$ would not be defined.

According to Corollary 1.2.3 we can state that the numbers $n_0$ prisotto of $N_0$ , subtracted from $N_0$ , result in all prime numbers in the interval ] $p_{max}$ , $N_0$ ].

**Remark 1.2.4** *Obviously, with $N_0$, being equal, the difference between the set of incongruous numbers less than $N_0$ modulo $\mathbb{P}\left(\sqrt{(\boldsymbol{N_0 - n_0})}\right)$ and that of the numbers prisotto of $N_0$ is given by all $n_0 = N - p_{0\,i}$ with $p_j \in \mathbb{P}\left(\sqrt{(\boldsymbol{N_0})}\right)$. In practice, the number of all odd primes less than or equal to $N_0$ is equal to the sum of the number of prisotto numbers of $N_0$ and that of $p_j \in \mathbb{P}\left(\sqrt{(\boldsymbol{N_0})}\right)$.*

**Remark 1.2.5** *Both theorem (1.2.1) and corollary (1.2.3) tell us nothing about the existence of at least one incongruous $n_0$ . However, according to postulate [6.3 of (b)] of Bertrand (later proved by Pafnuty Chebyshev, Srinivasa Ramanujan and Paul Erdős), which states that for each $n \geq 2$ there exists at least one prime p such that $n < p < 2n$, we can state, with respect to the corollary (1.2.3), that in the interval ] $p_{max}$ , $N_0$ ] there will always exist at least one prime being $2\,p_{max} \leq 2\sqrt{N_0} \leq N_0$ for $N_0 \geq 4$. Consequently, in the interval ]0, $N_0 - p_{max}$ [ there will always exist at least one $n_0$ prisotto of $N_0$.*

## 1.3 The Compcongruence of Natural Numbers

We now introduce **Complementary Congruence** (compcongruence) modulus m as the correspondence relation defined on the set of integers Z as follows: if m is a fixed integer greater than 1, two integers a and b are said to be compcongruent modulus m if $m|(a + b)$; m is called the modulus of the compcongruence and we will denote it by a ‖ b (mod m).

In the field of natural numbers, one can also equivalently state that a ‖ b (mod m) if a and b give two complementary remainders with respect to m in the integer division by m. For example, 24 ‖ 39 (mod 7) because they give as remainders in the integer division by 7 respectively 3 and 4, i.e. two complementary numbers with respect to 7.

## 1.4 Compcongruence Primality Theorem

**Enunciation 1.4.1** *$\forall N_0, n_0 \in N$ with $N_0 \geq 2$, $0 \leq n_0 \leq N_0 - 1$ and even if $N_0$ is odd or vice versa, with $\mathbb{P}\left(\sqrt{(N_0 + n_0)}\right)$ set of odd prime numbers $\leq \sqrt{(N_0 + n_0)}$, a necessary and sufficient condition for $N_0 + n_0$ to be a prime number is that $n_0 \nparallel N_0 \,(mod\, p_i) \; \forall p_i \in \mathbb{P}\left(\sqrt{(N_0 + n_0)}\right)$ or that $\mathbb{P}\left(\sqrt{(N_0 + n_0)}\right)$ is an empty set.*

**Proof.** According to the compcongruence of natural numbers (1.3), if $N_0$ and $n_0$ are **not** compcongruent modulo $p_i$ for all $p_i$ belonging to the ***set*** $\mathbb{P}\left(\sqrt{(\boldsymbol{N_0 + n_0})}\right)$ , this means that $N_0 + n_0$ is not divisible by any prime number less than the $\sqrt{(\boldsymbol{N_0 + n_0})}$ and that therefore, according to observation (1.1.2), $N_0 + n_0$ is a prime number. If, on the other hand $\mathbb{P}\left(\sqrt{(\boldsymbol{N_0 + n_0})}\right)$ is an empty set, the number $N_0 + n_0$ cannot be divided by any prime and is therefore prime.
Conversely, if $N_0 + n_0$ is a prime number, it will not be divisible by any other lower, equal or non-existent odd prime number of the $\sqrt{(N_0 + n_0)}$ and therefore $N_0$ and $n_0$ will always be non compcongrui $\forall p_i \in \mathbb{P}\left(\sqrt{(N_0 + n_0)}\right)$.

We set $N_0 \geq 2$ because with $N_0 = 1$ and $n_0 = 0$ we would have $N + n_{00} = 1$, a non-prime and non-compound number.

**Remark 1.4.2** *If, instead of referring to the set $\mathbb{P}(\sqrt{(\boldsymbol{N_0}+\boldsymbol{n_0})})$ we want to refer, for the needs of successive demonstrations, to the set $\mathbb{P}(\sqrt{\boldsymbol{2N_0}})$ the theorem (1.4.1) is transformed into the corollary (1.4.3).*

That is, given two numbers $N_0$ , $n_0$ ∈ N, with $n_0 < N_0$ and such that ($N_0$ +$n_0$ ) is odd, if it turns out that every odd prime number $p \leq \sqrt{(2N_0)}$ does not divide the number ($N_0$ +$n_0$ ) it means that it is prime.

Given a number $N_0$ ∈ N, a number $n_0$ ∈ N, less than or equal to $N_0$ and such that (N +$n_{00}$ ) is odd is called the **Prisopra of $N_0$** , if it turns out that $n_0 \not\equiv N_0 \pmod{p_i}$ ∀ $p_i \in \mathbb{P}(\sqrt{(2N_0)})$.

**Corollary 1.4.3** *∀ $N_0$ , $n_0$∈ N with $N_0 \geq 2$, $0 \leq n_0 \leq N_0 - 1$ and even if $N_0$ is odd or vice versa, with $\mathbb{P}(\sqrt{(2N_0)})$ set of odd prime numbers $\leq \sqrt{(2N_0)}$, a necessary and sufficient condition for $N_0$ + $n_0$ to be a prime number is that $n_0$ is a prisopra of $N_0$ .*

**Proof.** Extending the set of prime numbers of theorem (1.4.1) from $\mathbb{P}(\sqrt{(\boldsymbol{N_0}+\boldsymbol{n_0})})$ **a** $\mathbb{P}(\sqrt{(\boldsymbol{2N_0})})$ and indicating by $\mathbb{P}(\Delta 2N_0)$ the set of primes in $\mathbb{P}(\sqrt{(2N_0)})$ e non in $\mathbb{P}(\sqrt{(N_0+n_0)})$nothing changes since for each of the numbers $n_0$ (incompcongrui con $N_0$ moduli $\mathbb{P}(\sqrt{(N_0+n_0)})$
tali che $N_0$+$n_0$= $p_j$, with $p_j$ belonging to the interval [$N_0$ ,2$N_0$ ], it can never be the case that $n_0$ is compcongruent with $N_0$ modules $\mathbb{P}(\Delta 2N_0)$, e cioè che $[N_0]_{mod\ p_i}$+ $[n_0]_{mod\ p_i}$= $[0]_{mod\ p_i}$, for at least one $p_i \in$ $\mathbb{P}(\Delta 2N_0)$ **.** In fact, bearing in mind that $\sqrt{2N_0} \leq N_0$ con $N_0 \geq 2$ and that therefore all the primes $p_i$ belonging to the set $\mathbb{P}(\Delta 2N_0)$ are ≤ $N_0$ we have that for each $p_j$ belonging to the interval [$N_0$ ,2$N_0$ ] results $[p_j]_{mod\ p_i} \neq [0]$ always being $p_i$ *e* $p_j$ two prime numbers and different from each other. Consequently for each of the numbers $n_0$ tali che $N_0$+$n_0$= $p_j$, since modular arithmetic always results in [$N_0$ ] mod $p_j$ + [$n_0$ ] mod $p_j$ = $[p_j]_{mod\ p_i}$ and the latter is always different from zero, it can be stated that $n_0$ is prisopra of $N_0$ .
Conversely, if $N_0$ + $n_0$ is a prime number, it will not be divisible by any other lower, equal or non-existent odd prime number of the $\sqrt{(2N_0)}$ and therefore $N_0$ and $n_0$ will always be non compcongrui $\forall p_i \in \mathbb{P}(\sqrt{(2N_0)})$.

**Remark 1.4.4** *Both the theorem (1.4.1) and the corollary (1.4.3) tell us nothing about the existence of at least one $n_0$ prisopra of $N_0$ . But on the basis of Bertrand's postulate [6.3 of (b)] we can state that in the interval [$N_0$ , 2$N_0$ ] there will always exist at least one prime and consequently in the interval ]0, $N_0$ ] there will always exist at least one $n_0$ prisopra of $N_0$.*

### 1.5 Numbers and their congruence classes

Number Theory tells us that just as there exists in positional number systems (e.g. the decimal system) a bi-univocal correspondence between all possible numbers expressible with n digits (and therefore belonging to the interval ]0, $10^n$ -1]) and all possible combinations ($10^n$ ) of the 10 digits, similarly there exists a bi-univocal correspondence between all possible numbers of the interval ]0, $p_{max}$#], with $p_{max}$ any prime and $p_{max}$ # its prime, and the combinations of the congruence classes of these numbers having for modulus the single primes less than and equal to $p_{max}$ . The existence of this biunivocal correspondence is easily proved by resorting to the Chinese Remainder Theorem [2.3.3 of (b)] and inserting as modules of the system of equations $p_{max}$ and all primes less than it.

**Remark 1.5.1** We will call the **table number-classes $\boldsymbol{p_{max}}$** the table which, for each number in the interval ]0, $p_{max}$#] associates the combination of the congruence classes of this number having for modulus the single prime numbers less than and equal to $p_{max}$ .

For illustrative purposes, let us consider (see Appendix A) a **number-class table** 7 containing for each number the corresponding combination of its 4 congruence classes mod 2, mod 3, mod 5 and mod 7.
The above-mentioned bi-univocal correspondence can be verified in this table. E.g. the combination 1-2-2-3 of the congruence classes mod 2, mod 3, mod 5 and mod 7 corresponds only to the number 17 in the interval [1, 210], just as the number 151 corresponds only to the combination 1-1-1-4 of the same congruence classes in the interval [1, 210].

**As we shall see later, the Number-Class Table $p_{max}$ is introduced in this study in order to calculate the densities of the numbers prisotto and prisopra of $N_0$ .**

### 1.6 From the Number-Class Table $p_{max}$ to Primalities

Is there a criterion for deducing from the number-class table $p_{max}$ and from the information contained therein how many, in addition to the modules {2,3,.....,. $p_{max}$ } on which the table is built, are the prime numbers less than or equal to any $N_0 \in ]0, p_{max}\#]$ and those within the interval [$N_0$ , 2 $N_0$ ] ?

**Remark 1.6.1** *Among the various possible criteria, the one that interests us for our subsequent demonstrations consists in the application of the corollary (1.2.3) of the Primality of Congruence and that (1.4.3) of the Primality of Compongruence according to which the number of the odd primes less than or equal to $N_0$ is, minus the primes less than the $\sqrt{(\boldsymbol{N_0})}$ e cioè i modules {2,3,.....,. $p_{max}$ } on which the table is constructed, is equal to that of the numbers in the table prisotto of $N_0$ while the number of primes in the interval* [$N_0$ , 2 $N_0$ ] is equal to that of the numbers prisopra of $N_0$.

From this observation, it follows that in order to derive from the number-class table $p_{max}$ the prime numbers less than or equal to $N_0$ using the Primality of Congruence criterion, a condition must be imposed that binds $N_0$ to the number-class table $p_{max}$ and that is that the table's modules must be exactly all the primes less than or equal to the $\sqrt{(N_0)}$.

In the case of our example table [1, 210] we can say that only for the $N_0$ such that $7 \le \sqrt{(N_0)} < 11$, i.e. for $N_0$ greater than or equal to 49 and less than 121 we can say that the numbers in the table $n_0$ prisotto of $N_0$ are such that $N_0 - n_0$ is a prime number.

Similarly, to infer from the table-interval $]0, p_{max}\#]$ and from the information contained therein how many prime numbers there are in the interval [$N_0$ , $2N_0$ ] with $N_0 \in ]0, p_{max}\#]$ using the criterion (Corollary 1.4.3) of the Primality of Compcongruence we must impose that the modules {2,3,.....,. $p_{max}$ } of the table are exactly all the primes less than or equal to the $\sqrt{(2N_0)}$. With this condition we will have that the numbers of the table incompcongruent less than $N_0$ are pri $N_0$ such that such that $N_0 + n_0$ is a prime number.

In the case of our table [1, 210] for example we can say that only for the $N_0$ such that $7 \le \sqrt{(2N_0)} <$ 11 and i.e. for $N_0$ greater than or equal to 25 and less than 61 we can say that the numbers $n_0$ incompcongruous less than $N_0$ are prisopra of $N_0$ and i.e. that added to $N_0$ give the prime numbers of the interval [$N_0$ , $2N_0$ ].

### 1.7 From $N_0$ to the primes of the interval $]0, 2N_0]$

If then, fixed at any $N_0 \in N$ greater than 49, we want to find out how many prime numbers are less than or equal to $N_0$ we must first find the highest prime number $p_{max}$ less than or equal to $\sqrt{(N_0)}$ and then consider the number-class table $p_{max}$ $]0, p_{max}\#]$, where $p_{max}\#$ is the prime of $p_{max}$ and corresponds to the product of prime numbers $\le p_{max}$ *Since the prime $p_{max}\#$ coincides with the prime $\sqrt{(\boldsymbol{N_0})}\ \#$ in the remainder of the study we will write either $]0, p_{max}\#]$ o $]0, \sqrt{(\boldsymbol{N_0})}\ \#]$ to indicate the* *same* Number-class table $p_{max}$ .

**Remark 1.7.1** *The condition that $N_0$ is greater than or equal to 49 follows from the requirement that $N_0$ belongs to the interval ]0, $p_{max}$#].*

From what is written in observation (1.6.1) the number of primes less than or equal to $N_0$ is given to us, minus the primes less than the $\sqrt{(\boldsymbol{N_0})}$ e cioè i forms 2, 3, ....., $p_{max}$ on which the table is constructed, by that of the numbers in the table prisotto of $N_0$ , a number which, according to observation (1.2.5) will always be equal to or greater than 1.

E.g. with $N_0$ = 315 we will have that $\sqrt{315}$=17.746 and therefore $p_{max}$ will be equal to 17, $p_{max}$# (2*3*5*7*11*13*17) will equal 510510 and the number 315 will correspond, in the interval [0 , $p_{max}$#], one and only one combination of its congruence classes mod 2, mod 3, mod 5, mod 7, mod 11, mod 13 and mod 17. All $n_0$ less than $N_0$ and incongruent with it with respect to $p_i \leq p_{max}$i.e. all $n_0$ prisotto of $\mathrm{N_0}$, sottratti ad $\mathrm{N_0}$ result in all prime numbers less than $N_0$ , except the primes 2,3,5,7,11,13,17 on which the table is built. On the other hand, according to the corollary (1.2.3) and observation (1.6.1), nothing can be said about the other numbers della tabella $\mathrm{m_0}$ maggiori di 315 and incongruous with it p modules$_i$ belonging to $\mathbb{P}\left(\sqrt{(315)}\right)$.

Similarly, if we want to find, via a number table $p_{max}$how many prime numbers there are in the interval [$N_0$ , $2N_0$ ] with any $N_0 \geq 121$, we must first find the highest prime number $p_{max}$ less than the $\sqrt{2N_0}$ and then consider the number-class table $p_{max}$ ]0, $\sqrt{(2\boldsymbol{N_0})}$#]. Again, the condition that $N_0$ is greater than or equal to 121 follows from the requirement that $2N_0$ belongs to the interval ]0, $\sqrt{(2\boldsymbol{N_0})}$#]. According to observation (1.6.1), the number of primes in the interval [$N_0$, $2N_0$ ] is given to us by that of the numbers in the table prisopra of $N_0$ , a number which according to observation (1.4.4) will always be equal to or greater than 1.

If we maintain the previous example of $N_0$ = 315, we must in this case calculate the $\sqrt{2 * 315}$ which is 25.1, from which it follows that $p_{max}$ will be equal to 23, $\sqrt{(2\boldsymbol{N_0})}$# (equal to 2*3*5*7*11*13*17*19*23) will be equal to 223092870 and the number 315 will correspond, in the interval ]0, $\sqrt{(2\boldsymbol{N_0})}$#], one and only one combination of its congruence classes mod 2, mod 3, mod 5, mod 7, mod 11, mod 13, mod 17, mod 19, mod 23. All $n_0$ less than $N_0$ and incompcongruent with it, i.e. all $n_0$ prisopra of $\mathrm{N_0}$, added to$\mathrm{N_0}$ will result in all prime numbers in the interval [$N_0$ **,** **2** $N_0$ ]. On the other hand, according to the corollary (1.4.3) and observation (1.6.1), nothing can be said about the other numbers della tabella $\mathrm{m_0}$ greater than 315 and incompcongruent with it modules $\mathbb{P}\left(\sqrt{(315)}\right)$.

## 2 The distribution of prime numbers

### 2.1 Fundamental prime number theorem

Gauss's Conjecture, dating back to 1792 and later becoming the Prime Number Theorem (NPT), on the distribution of prime numbers is:

(2.1.1) $\pi(N) \approx \frac{N}{logN} \approx \int_2^N \frac{dt}{logt} \approx Li(N)$

where π(N) is the number of primes less than or equal to N.

This conjecture was first proved in 1986 by Hadamard and de La Vallée Poussin using methods from the theory of complex functions related to the properties of Riemann's ζ-function. Mathematicians of the time, and in particular G. H. Hardy, believed that complex analysis was necessarily involved in the Theorem and that methods with only real variables were to be considered inadequate. But in 1949, Erdös and Selberg [3.4 of (a)] independently published an elementary proof (i.e. with only real variables), based on the combinatorial technique, of the Prime Number Theorem.

The demonstration of Selberg - Erdős [3.4 of (a)] thus brought into play the supposed superiority (depth) of complex analysis for the demonstration of NPT, showing that even technically elementary methods, which we have also adopted in this study, have their demonstrative effectiveness.

### 2.2 The average density of the $n_0$ incongruous of $N_0$ in the table ]0, $\sqrt{(N_0)}$#]

Having fixed any $\boldsymbol{N_0} \in \boldsymbol{N}$ greater than 49, we consider (see paras. 1.6 and 1.7) the relevant number-class table $p_{max}$ of the interval ]0, $p_{max}$ #], where $p_{max}$ is the highest prime number less than or equal to the $\sqrt{(N_0)}$, and calculate the number of all (greater than and less than $N_0$) the $n_0$ incongruous of $N_0$ present in the table.

We then eliminate from this table the rows that have one or more classes of congruence of the p modules$_i$ (2, 3, 5, ..... , $p_{max}$) equal to the class corresponding to the remainder of $N_0$ for the same modules.

The numbers M in the table, not eliminated through the previous sieve, can then only be those which in the number-class table $p_{max}$ have for each $p_i \in \mathbb{P}\left(\sqrt{(N_0)}\right)$ one of the $p_i$ -1 possible congruence classes other than the corresponding $N_0$ . (If e.g. ($N_0$ ) mod7 = 3, (M) mod7 must be equal to one of the 6 (7-1) other possible congruence classes: 0,1,2,4,5,6)

The rows of the table that have not been deleted will then, according to the combinatorial calculation, be:

(2.2.1) $\prod_{p=2}^{p_{max}}(p-1)$

Thus, (2.2.1) gives us the quantity of all **M** numbers in the table **incongruous (less than and greater than) than $N_0$ for the p modules only$_i$** belonging to the set $\mathbb{P}\left(\sqrt{(N_0)}\right)$.

Let us now calculate the **average density Dnc**$_{]0,\sqrt{N_0}\#]}$ of these numbers M existing in the interval ]0, $\sqrt{(N_0)}$ #] with $\sqrt{(N_0)}$ # = 2*3*.......*$p_{max}$can be written:

(2.2.2) $Dnc_{]0,\sqrt{N_0}\#]} = \frac{\prod_{p=2}^{p_{max}}(p-1)}{2*3*...*pmax} = \frac{\prod_{p=2}^{p_{max}}(p-1)}{\prod_{p=2}^{p_{max}}p} = \prod_{p=2}^{p_{max}}\frac{(p-1)}{p}$

[*formula this multiplied by* $\sqrt{(N_0)}$ # *corresponds to the Euler function φ(n) with n=*$\sqrt{(N_0)}$ #*, and gives the number of coprimes less than* $\sqrt{(N_0)}$ #*, a number which also includes the number of primes less than* $N_0$ *except for the primes belonging to the* set $\mathbb{P}\left(\sqrt{(N_0)}\right)$]

On the basis of the corollary (1.2.3) of the Primality of Congruence and the fact that all numbers M less than $N_0$ (**$M_{N0}$** ) are **prisotto of $N_0$**, we can state that, for each of these numbers $M_{N0}$ , $N_0$ -$M_{N0}$ is a prime number and that the average density $Dnc_{]0,N_0]}$ [1)] of M $_{N0}$ in the interval ]0, $N_0$ ] is given by**:**

**(2.2.3)** $Dnc_{]0,N_0]} = \frac{Q(M_{N0})}{N_0}$ denoting by Q(M )$_{N0}$ the number of $M_{N0}$ present in the interval ]0, $N_0$ ].

As per observation (1.2.4) the number of **all** primes $\pi(N)_0$ less than or equal to $N_0$ is given by the sum of the number of $M_{N0}$ and that of all $p_j \in \mathbb{P}(\sqrt{(N_0)})$ which, as we know, are not among the $N_0 - M_{N0}$ .

We then know from NPT (2.1) that the average density $Dprimi_{N0}$ of the prime numbers less than $N_0$ , which coincides, barring the $p_i$ belonging to the ***set*** $\mathbb{P}(\sqrt{(N_0)})$, with the average density $Dnc_{N0}$ of the numbers $M_{N0}$ **prisotto of $N_0$** is given by:

$$(2.2.4)\quad Dprimi_{]0,N_0]} = \frac{\pi(N_0)}{N_0} = \frac{1}{logN_0} \approx Dnc_{]0,N_0]}$$

That is, for the density $Dprimi_{N0}$ one must consider, in addition to the numbers $M_{N0}$ less than $N_0$ and incongruous with it, also the pi belonging to the set $\mathbb{P}(\sqrt{(N_0)})$ and consequently *Dprimi* always results$_{N0} > Dnc_{N0}$. Let us then calculate the error that is made by setting Dprimi = $Dnc_{N0}$ . According to NPT (2.1) we can write:

$$(\,2.2.5\,)\; Dnc_{]0,N_0]} = \frac{\left(\frac{N_0}{logN_0} - \frac{\sqrt{N_0}}{log\sqrt{N_0}}\right)}{N_0} \qquad \text{e } Dprimi_{]0,N_0]} = \frac{1}{logN_0}$$

**Observation 2.2.6** *Having ascertained that it always results $Dprimi_{]0,N_0]} > Dnc_{]0,N_0]}$ one can easily calculate that the percentage error in positing $Dprimi_{]0,N_0]} = Dnc_{]0,N_0]}$ is 20% for $N_0 = 10^2$ , 2% for $N_0 = 10^4$ , 0.02% for $N_0 = 10^8$ , and that it is gradually decreasing for increasing values of $N_0$ .*

## 3 The Proof of the Hardy-Littlewood Conjecture

### 3.1 The Twin Numbers

Every prime number greater than 2, likewise every odd number, can be written as the sum or difference of an even number and 1. In the case of a pair of prime twins there will obviously be a single even number which when added to 1 and subtracted by 1 will give rise to the prime twins of the pair.

We call an even twin and denote by the symbol PG any even number n ∈ N such that n+1 and n-1 are two prime numbers.

### 3.2 The Equal Twins Theorem

**Definition 3.2.1** *$\forall\, n_0 \in N$, even and greater than 4, with $\mathbb{P}(\sqrt{(n_0+1)})$ set of odd primes $\le \sqrt{(n_0+1)}$, a necessary and sufficient condition for $n_0$ +1 and $n_0$ -1 to be twin primes is that $n_0 \not\equiv 1(mod\, p_i)$ and $n_0 \not\parallel 1(mod\, p_i)$ $\forall p_i \in \mathbb{P}(\sqrt{(n_0+1)})$ or that $\mathbb{P}(\sqrt{(n_0+1)})$ is an empty set.*

**PROOF.** From the two Primality Theorems (1.2.1) and (1.4.1), assuming $N_0 = n_0$ and $n_0 = 1$ it follows that, if 1 is incongruous and incompcongruous with $n_0$ p modules$_i$ $\forall p_i \in \mathbb{P}(\sqrt{(n_0+1)})$, e di conseguenza $\forall p_i \in \mathbb{P}(\sqrt{(n_0-1)})$ essendo $\mathbb{P}(\sqrt{(n_0-1)}) \subseteq \mathbb{P}(\sqrt{(n_0+1)})$, or if $\mathbb{P}(\sqrt{(n_0+1)})$ is an empty set, $n_0$ +1 and $n_0$ - 1 are twin primes.
Conversely, if $n_0$ + 1 and $n_0$ - 1 are twin primes, this means that they are not divisible by any prime less than or equal to the $\sqrt{(n_0+1)}$ and that therefore, again by (1.2.1) and (1.4.1), $n_0$ and 1 are incongruous and incompcongruous $\forall p_i \in \mathbb{P}(\sqrt{(n_0+1)})$ and therefore $\forall p_i \in \mathbb{P}(\sqrt{(n_0-1)})$.

We set $n_0 \geq 4$ because with $n_0 = 2$ we would have that $n_0 - 1 = 1$ which, as we know, is neither a prime nor a compound number.

If instead of referring to the set $\mathbb{P}\left(\sqrt{(n_0+1)}\right)$ we refer, for the sake of subsequent demonstrations, to the set $\mathbb{P}\left(\sqrt{N_0}\right)$ with $N_0 \in N$ e maggiore di $n_0$ the theorem (3.2.1) is transformed into the following corollary:

**Corollary 3.2.2** *∀ $N_0$, $n_0 \in N$, with $N_0 \geq 9$ and with $n_0$ even and $p_{max} < n_0 < N_0$, with $\mathbb{P}\left(\sqrt{N_0}\right)$ set of odd prime numbers $\leq \sqrt{(N_0)}$ and with $p_{max}$ higher prime number than $\mathbb{P}\left(\sqrt{N_0}\right)$, a necessary and sufficient condition for $n_0$ +1 and $n_0$ -1 to be twin primes is that 1 be an incongruous and incompcongruous number of $n_0$ .*

**Proof.** substituting $\mathbb{P}\left(\sqrt{(N_0)}\right)$ a $\mathbb{P}\left(\sqrt{(n_0+1)}\right)$ the numbers $n_0$ even less than $p_{max}$ and such that $n_0 \pm 1 = p_j$ , with $p_j \in \mathbb{P}\left(\sqrt{(N_0)}\right)$, are not taken into account since, for the same $p_j \in \mathbb{P}\left(\sqrt{(N_0)}\right)$, they have a congruence class mod $p_j$ equal and/or complementary to that of equal modulus of 1. In fact, if $n_0 \pm 1 = p_j$ according to modular arithmetic it will always be the case that $[n_0]$ mod $p_j \pm [1]$ mod $p_j$ = $[p_j]$ mod $p_j$ = [0] from which the congruence and/or compcongruence mod $p_j$ of 1 with $n_0$ follows.
Conversely, if $n_0$ + 1 and $n_0$ - 1 are twin primes greater than $p_{max}$ and less than $N_0$ it means both that, according to (1.2.1) and (1.4.1), $n_0$ and 1 are incongruous and incompcongruous $\forall p_i \in \mathbb{P}\left(\sqrt{(n_0 \pm 1)}\right)$, but also that, since $n_0$ + 1 and $n_0$ - 1, as primes, are not divisible by any prime less than or equal to $\sqrt{(N_0)}$ , $n_0$ and 1 are incongruous and incompcongruous anche $\forall p_i \in \mathbb{P}\left(\sqrt{(N_0)}\right)$.
He placed himself $N_0 \geq 9$ in quanto per valori inferiori $p_{max}$ would not be defined.

Since in the interval $]0, N_0]$, with $N_0 \geq 9$ and $n_0 > p_{max}$ there is always at least one prime (observation 1.2.5), surely there will always exist an $n_{01}$ and an $n_{02}$ of which 1 is incongruous and incompcongruous; but in order to prove the Hardy-Littlewood conjecture [1 of (c)] we must ascertain both that there exists at least one $n = n =_{0\ 01\ \ 02}$ , ni.e. an even twin number (PG), smaller than $N_0$ , of which 1 is incongruous and incompcongruous modulo $p_i$ for all $p_i$ belonging to the ***set*** $\mathbb{P}\left(\sqrt{(\boldsymbol{N_0})}\right)$, let it be that for $N_0 \rightarrow \infty$ the number of PGs also tends to infinity with a definite relation.
To this end, we resort to the study of the density of twin peers.

### 3.3 The density of twin peers

All $n_0$ that satisfy the conditions of corollary (3.2.2) are even PG twins with the following characteristics:

- the class of PG module 2, PG being even, is always zero while the class of 1 module 2 is always 1 (with complement equal to 1) and consequently 1 will always be incongruous and incompcongruous with PG module 2
- PG classes of the next module (3, 5, 7, 11, etc.) present in $\mathbb{P}\left(\sqrt{(N_0)}\right)$ must not be equal to the classes of 1 and their p-1 complements of the same module (*e.g. if PG=18 and* $N_0 = 24$ *we have that* $\mathbb{P}(\sqrt{N_0})$ *= {3}; $[18]_{mod3}$ = 0 and its complement is still equal to 0, $[1]_{mod3}$ =[1] and its complement is equal to 2 and therefore 1 is incongruous and incompcongruous with PG so that 18+1 and 18-1 are twin primes*).

Having said this, let us see how to calculate the number of PGs and thus of pairs of twins less than a $N_0 \geq 49$, a condition (see observation 1.7.1) arising from the necessity that $N_0$ belongs to the interval $]0, p_{max}\#]$ where $p_{max}$ is the highest prime number less than or equal to the $\sqrt{(N_0)}$ (1.5).

Having then selected any $N_0 \geq 49$ we denote by $p_{max}$ the highest prime number of $\mathbb{P}(\sqrt{(N_0)})$. Let us then consider the interval/table of natural numbers $]0, p_{max}\#]$ and now eliminate from this table the rows that have: congruence class mod 2 equal to [1]; congruence classes of successive modules (3, 5, ..... , $p_{max}$ ) equal to the classes of 1 and their complements p-1 for the same modules.
The numbers **M of** the number-classes table $p_{max}$ , not eliminated through the previous sieve, can then only be those which in their corresponding combination of congruence classes present only the class [0] of the two possible congruence classes mod 2 and one of the $p_i$ -2 (for each $p_i$ belonging to the set $\mathbb{P}(\sqrt{(N_0)})$) possible classes of congruence of the successive modules (3, 5, ..... , $p_{max}$ ), that is, with the exclusion of the classes of 1 and their complements for the same modules (if e.g. (M) mod7 = 1 with complement = 6, M will not be a twin pair since (1) mod7=1 with complement = 6; to be a twin pair it is necessary that (M) mod7 is equal to one of the 5 (7-2) possible other classes of congruence: 0,2,3,4,5)

The rows (combinations of classes) of the table that have not been deleted will then, according to combinatorial calculation, be:

(3.3.1) $\prod_{p=3}^{p_{max}}(p-2)$

Thus, (3.3.1) gives us the quantity of the numbers **M of** the table-interval $]0, p_{max}\#]$ of which 1 **is not congruent and is not compcongruent only for the modules $p_i$** belonging to the set $\mathbb{P}(\sqrt{(N_0)})$ and nothing can be said about the possible (non) congruence and (non) compcongruence of 1 with these numbers with respect to the other modules $p_j$ greater than $p_{max}$ and belonging to the set $\mathbb{P}(\sqrt{(pmax\#)})$). On the basis of the corollary (3.2.2) we can then state that all numbers **M(PG) less than $N_0$** , being non-congruent and non-compcongruent with 1 for all modules $p_i$ belonging to the set $\mathbb{P}(\sqrt{(N_0)})$, are even twin numbers (PG).

**Remark 3.3.2** *By the same corollary (3.2.2) we also know, however, that these numbers M(PG), of which 1 is not compcongruent with respect to the modules* $\mathbb{P}(\sqrt{(N_0)})$*, do not include the PGs relative to the pairs of primes less than the* $\sqrt{N_0}$ *and consequently their average density* $Dncncomp(PG)_{N0}$ *will always be lower than that* $Dpg_{N0}$ *of all pairs of P twins*$G_{N0}$ *less than* $N_0$ .

We now calculate the average density $Dncncomp_{]0,\ p_{max}\#]}$ of the PG numbers existing in the interval $]0, p_{max}\#]$ of which 1 is non-congruent for the p-modules$_i$ belonging to the set $\mathbb{P}(\sqrt{(N_0)})$. Knowing that $p_{max}\# = 2*3*.......*p_{max}$ , we can write:

(3.3.3) $Dncncomp_{]0,\ p_{max}\#]} = \frac{\prod_{p=3}^{pmax}(p-2)}{\prod_{p=2}^{pmax}p} = \frac{1}{2} * \prod_{p=3}^{pmax}\frac{(p-2)}{p}$

multiplying and dividing the second term of the same by (p-1) we obtain:

(3.3.4) $Dncncomp_{]0,\ p_{max}\#]} = \frac{1}{2} * \prod_{p=3}^{pmax}\frac{(p-2)}{p} * \frac{(p-1)}{(p-1)} = \frac{1}{2} * \prod_{p=3}^{pmax}\frac{(p-1)}{p} * \prod_{p=3}^{pmax}\frac{(p-2)}{(p-1)} = \prod_{p=2}^{pmax}\frac{(p-1)}{p} * \prod_{p=3}^{pmax}\frac{(p-2)}{(p-1)}$

In the last member of (3.3.4) we have substituted for $\frac{1}{2} * \prod_{p=3}^{pmax}\frac{(p-1)}{p}$ the term $\prod_{p=2}^{pmax}\frac{(p-1)}{p}$ which, as we know from (2.2.2), corresponds, always for $N_0 \geq 49$, to the average density $Dnc_{]0,\ \sqrt{N_0}\#]}$ of the numbers M existing in the interval $]0, p_{max}\#]$ **not congruent of $N_0$ for only the modules $p_i$** belonging to the set $\mathbb{P}(\sqrt{(N_0)})$; in these last three formulae $p_{max}$ is the highest prime number less than or equal to the $\sqrt{(N_0)}$ .

Let us then see if we can find a relationship between $\prod_{p=3}^{pmax}\frac{(p-2)}{(p-1)}$ e $\prod_{p=2}^{pmax}\frac{(p-1)}{p}$ so that we can determine the value of $Dncncomp_{]0,\ p_{max}\#]}$ as a function of $Dnc_{]0,\ \sqrt{N}_0\#]}$.

We can write:

(3.3.5) $\frac{\prod_{p=3}^{pmax}\frac{(p-2)}{(p-1)}}{\prod_{p=2}^{pmax}\frac{(p-1)}{p}} = \prod_{p=3}^{pmax}\frac{(p-2)}{(p-1)} * \prod_{p=2}^{pmax}\frac{p}{(p-1)} = \prod_{p=3}^{pmax}\frac{(p-2)}{(p-1)} * 2 * \prod_{p=3}^{pmax}\frac{p}{(p-1)} = 2 * \prod_{p=3}^{pmax}\frac{p*(p-2)}{(p-1)^2}$

where one can easily verify that the relationship between the term $\prod_{p=3}^{pmax}\frac{(p-2)}{(p-1)}$ and that $\prod_{p=2}^{pmax}\frac{(p-1)}{p}$ for $N_0$ =49 takes on the value 0.68359375, for $N_0$ =9006001 the value 0.6601862196 and then, as N increases$_0$ towards infinity, and thus extending the product over all prime numbers $\geq 3$, it tends rapidly to decrease towards the constant of the prime twins $C_2$ that appears in the Hardy-Littlewood conjecture on the distribution of prime twins:

$\prod_{p\geq 3}\frac{p*(p-2)}{(p-1)^2}$ = $C_2 \approx$ 0.6601611813846869573927812110014 ..........

We can therefore write:

(3.3.6) $\prod_{p=3}^{pmax}\frac{(p-2)}{(p-1)} \approx 2*C*_2\prod_{p=2}^{pmax}\frac{(p-1)}{p}$

which substituted in (3.3.4) gives us:

$Dncncomp(PG) \approx \prod_{p=2}^{pmax}\frac{(p-1)}{p} * \prod_{p=3}^{pmax}\frac{(p-2)}{(p-1)} \approx 2*C*_2(\prod_{p=2}^{pmax}\frac{(p-1)}{p})^2$

from which according to (2.2.2):

(3.3.7) $Dncncomp_{(PG)]0,\ p_{max}\#]} \approx 2*C_2*(Dnc_{]0,\ \sqrt{N}_0\#]})^2$

In this relation $Dncncomp_{(PG)]0,\ p_{max}\#]}$ to the square of Dnc the constant $C_2$ changes little as N varies$_0$ and thus as p varies$_{max}$ #. The sieve that determines the density $Dncncomp_{(PG)]0,\ p_{max}\#]}$ in fact depends neither on $N_0$ nor on $p_{max}$ # but only on the incongruity and incompcongruity of 1 with an $n_0$ =PG $\forall p_i \in \mathbb{P}(\sqrt{(n_0 \pm 1)})$.
It can therefore be written with good approximation:

( 3.3. 8) $Dncncomp_{]0,\ N_0]}\ {}_2 * 2*C \approx (Dnc_{]0, N_0]]})^2$

**Comment 3.3.9** *In (3.3.8) as stated in Comments (3.3.2) and (2.2.6) both* $Dncncomp_{]0,\ N_0]}$ *and* $Dnc_{]0, N_0]]}$ *do not include the possible $n_0$ for which $n_0 \pm 1$ are equal to the primes less than or equal to the $\sqrt{(N_0)}$ but since this relation is always valid $\forall N_0 \in N$ starting from $N_0$ = 49 we can extend (3.3.8) to all the numbers $n_0$ of which 1 is not congruous and not compcongruous and which when added to or subtracted from 1 result in the primes (except 2, 3, 5, 7) less than any $N_0$ greater than 49. In fact for $N_0$ = 49 (and therefore $\sqrt{49}$ = 7) the (3.3.8)* concerns all the $n_0$ of which 1 is *not congruous and not compcongruous* that subtracted and added to 1 give as result the first twins *between 8 and 49; for $N_0$ = 121 (and therefore $\sqrt{121}$ = 11), (3.3.8) concerns the first twins between 12 and 121;*

*for $N_0$ = 169 (and thus $\sqrt{169}$ = 13), (3.3.8) concerns the first twins between 14 and 169; and we can continue in this way for all subsequent $N_0$ equal to the squares of the first twins after 13.*

But it can be verified, assuming $N_0$ =49 and thus $C_2$ =0.6835, that (3.3.8) with an approximation of about 5%, also subsists taking into account primes 2, 3, 5, 7. In fact with $N_0$ =49 we count 15 primes and 6 even twins whence (3.3.8) becomes:

$$\frac{6}{49} \approx 2*0{,}6835*\left(\frac{15}{49}\right)^2$$

0,1224 ≈ 0,1281

Obviously, as N increases$_0$ , subject to the validity of (3.3.8) for all primes greater than 7, the approximation decreases.
Ultimately, we can then hold that ∀ $N_0$ ∈ N greater than 49, (3.3.8) is valid for all primes less than $N_0$ and therefore, substitute $Dpg_{N0}$ al posto di $Dncncomp(PG)_{(N0)}$ e $Dprimi_{]0,\,N_0]}$ al posto di $Dnc_{]0,\,N_0]}$, writing:

(3.3.10) $Dpg_{N0} \approx 2*C_2*(Dprimi_{]0,\,N_0]})^2$

Being then for the NPT $Dprimi_{]0,\,N_0]} = \frac{1}{logN_0}$ one can write:

(3.3.11) $Dpg_{N0} \approx 2*C_2*(\frac{1}{logN_0})^2$

and multiplying both members by $N_0$ :

(3.3.12) $PG_{N0} \approx N*2*C*_{02}(\frac{1}{logN_0})^2$

Appendix C gives an example of $n_0$ of which 1 is prisotto and prisopra and the corresponding values of $Dnc_{]0,\,N_0]}$, $Dncncomp_{]0,\,N_0]}$ and $PG_{NO}$ verified and calculated.

For $N_0$ = 49 the (3.3.12) $PG_{N0}$ takes a value greater than 5 and, since $N*_0(\frac{1}{logN_0})^2$ an increasing function with $N_0$ , $PG_{N0}$ will always grow as $N_0$ tends to infinity with a distribution **(**3.3.12**)** equal to that predicted by the Hardy-Littlewood conjecture [(c)]:

$$\pi_2(x) \approx x*2*C*_2(\frac{1}{logx})^2$$

The even twins, i.e. pairs of prime twins, are therefore infinite and (3.3.12) is their distribution law.

## 4 The Proof of the Goldbach Conjecture

Goldbach's conjecture assumes that for every even number $2N_0$ there exist one or more numbers n ∈ N such that $N_0$ - n and $N_0$ + n are two prime numbers whose sum is obviously equal to $2N_0$ .

Given an $N_0$ ∈ N we denote by the letter Ꞡ every number n ∈ N such that $N_0$ - n and $N_0$ + n are two prime numbers.

### 4.1 The Ꞡ Number Theorem of $N_0$

**Definition 4.1.1** *∀ N , $n_0$ ∈ N and $n_0$ even if $N_0$ is odd or vice versa, with $N_0 \geq 9$, $0 \leq n_0 \leq N_0 - p_{,max}$ with $p_{max}$ being a prime number higher than $\mathbb{P}(\sqrt{2N_0})$, where $\mathbb{P}(\sqrt{2N_0})$ è l' set of odd prime numbers $\leq \sqrt{(2N_0)}$, a necessary and sufficient condition for $N_0 - n_0$ and $N_0 + n_0$ to be two prime numbers is that $n_0$ is a prisotto number and prisopra of $N_0$ .*

**Proof.** From the Corollaries (1.2.3) and (1.4.3), placing the most restrictive conditions between the two, derive the necessary and sufficient conditions of the Theorem. Just as from Observations (1.2.5) and (1.4.4) it follows that there surely exists at least one $n_{01}$ prisotto and at least one $n_{02}$ prisopra of $N_0$ but we cannot derive from them that esiste anche un $n_0 = n_{01} = n_{02}$.
To prove Goldbach's conjecture, on the other hand, it must be established that for every $N_0 \geq 9$ there exists at least un $n_0 = n_{01} = n_{02}$ i.e. a number Ɠ, prisotto and prisopra of $N_0$ .

Apart from the special case of a prime $N_0$ and thus the certain existence of a Ɠ = 0, we must therefore prove that for every $N_0$ there always exists a Ɠ prisotto and prisopra of $N_0$ and thus that there always exist two prime numbers equidistant from $N_0$ :

$p_1 = N_0$ - Ɠ

$p_2 = N_0$ + Ɠ

and whose sum is evidently equal to $2N_0$ .

To this end, we resort to the study of the density of numbers Ɠ.

### 4.2 The density of numbers Ɠ

Let us say right away that each Ɠ must have the following characteristics:

- its class of modulus 2 must be equal to zero if $N_0$ is odd, to 1 if $N_0$ is even;
- its successive first module classes (3, 5, 7, etc.) less than or equal to the $(\sqrt{2N_0})$ must not be equal to the two classes corresponding to the remainder (for non-congruence) and its complement (for non-compcongruence) of $N_0$ for the same modules (*e.g. if* $N_0 = 43$ *and Ɠ=30 we have that* $\mathbb{P}(\sqrt{N_0})$ *= {3,5}; $[43]_{mod3}$ = 1 with a complement equal to 2, $[43]_{mod5}$ = 3 with a complement equal to 2; $[30]_{mod3}$ =[0] and $[30]_{mod5}$ =[0]; therefore Ɠ is prisotto and prisopra of $N_0$ and therefore 73 (43+30) and 13 (43-30) constitute a pair of primes whose sum is equal to $2N_0$ ).*

Having said this, let us see how to calculate the number of Ɠ less than an $N_0 \geq 121$ (a condition deriving as we know (1.7.1) from the need for $2N_0$ to belong to the interval $]0, \sqrt{(\boldsymbol{2N_0})}\#]$).
Having then selected any $N_0 \geq 121$, we call $p_{max}$ the highest prime number less than or equal to the $\sqrt{(\boldsymbol{2N_0})}$. Let us then consider the table-interval of natural numbers $]0, p_{max}\#]$ where $p_{max}\#$ is the prime of $p_{max}$ and corresponds to the product 2*3*5*.......* $p_{max}$ , a product that corresponds to the last number of the relevant Number-Class Table $p_{max}$ (1.5.1) of bi-univocal correspondence between the numbers of the interval and the respective combinations of their congruence classes.

Let us now eliminate from this table $]0, p_{max}\#]$ each of the rows that has a congruence class mod 2 equal to 0 or to 1 depending on whether $N_0$ is even or odd, and/or congruence classes of the following modules (3, 5, ..... , $p_{max}$ ) equal to one of the two classes corresponding to the remainder and complement of $N_0$ for the same modules.
The M-numbers in the table, which were not eliminated through the previous sieve, can then only be:

a) those which in the number-class table $p_{max}$ have in their corresponding combination of congruence classes only one of the two possible congruence classes modulo 2
b) those which in the number-class table $p_{max}$ for each odd $p_i$ belonging to the set $\mathbb{P}(\sqrt{(\mathbf{2N_0})})$ and NOT FACTOR of $N_0$ have in their corresponding combination of congruence classes one of the $p_i$ -2 possible congruence classes of the modules 3, 5, ..... , $p_{max}$ that is, with the exclusion of the two classes corresponding to the remainder and the complement of $N_0$ for the same modules $p_i$ (*if e.g. ($N_0$ ) mod7 = 3 with complement = 4, (M) mod7 must be equal to one of the 5 (7-2) other possible congruence classes: 0,1,2,5,6*)
c) those which in the number-class table $p_{max}$ for every odd $p_i$ belonging to the set $\mathbb{P}(\sqrt{(\mathbf{2N_0})})$ and FACTOR of $N_0$ have in their corresponding combination of congruence classes one of the $p_i$ -1 possible congruence classes other than [0] that constitutes both the remainder and the complement of $N_0$ for the same module-factors.

The numbers $N_0$ with factors other than $p_i$ odd belonging to the set $\mathbb{P}(\sqrt{(\mathbf{2N_0})})$and which therefore fall under category b) of the previous classification, are the prime numbers outside the set $\mathbb{P}(\sqrt{(\mathbf{2N_0})})$ or a multiple of them with coefficient $2^n$ or a simple power of 2.
In particular, let us consider only the prime numbers that we will call $N_{0pm}$ indicating by $\mathbb{P}$ their set.

For the numbers $N_{0pm}$ then the rows (combinations of classes) of the table ]0, $p_{max}$ #] not deleted, according to combinatorial calculation, will result to be:

(4.2.1) $\prod_{p=3}^{p_{max}}(p-2)$

(4.2.1) thus provides us with the quantity of numbers M in the table that are incongruent and incompcongruent with $N_{0pm}$ , while nothing can be said about their possible (non) congruence and/or (non) compcongruence with $N_{0pm}$ with respect to the other modules $p_j$ greater than $p_{max}$ and belonging to the set $\mathbb{P}(\sqrt{(\text{pmax\#})})$.

According then to the Ꞡ Number Theorem (4.1.1) we can state that all numbers **M less than $N_{0pm}$ ($M_{Ꞡ}$** ) are prisotto and prisopra to $N_{0pm}$ and are therefore numbers Ꞡ.

**Remark 4.2.2** *By the corollary (1.2.3) and remark 1.2.4 we also know, however, that such numbers $M_{Ꞡ}$ , (prisotto and prisopra of $N_{0pm}$ ) do not include the possible $n_0$ for which ($N_{0pm}$ - $n_0$ ) is equal to a $p_i$ belonging to the set $\mathbb{P}\left(\sqrt{(2N_{0pm})}\right)$. Consequently, all numbers Ꞡ less than $N_{0pm}$ are always greater than/equal to the numbers $M_{Ꞡ}$ .*

The *average* density $Dncncomp_{]0,\ p_{max}\ \#]}$ of the numbers M existing in the interval ]0, $\sqrt{(2\mathrm{N}_{0\mathrm{pm}})}$#] **uncongruent with $N_{0pm}$ for only p-modules$_i$** belonging to the set $\mathbb{P}\left(\sqrt{(2\mathrm{N}_{0\mathrm{pm}})}\right)$, knowing that $\sqrt{(2\mathrm{N}_{0\mathrm{pm}})}$# = 2*3*.......* $p_{max}$ , can be written:

(4.2.3) $Dncncomp_{]0,\ \sqrt{2N_{0pm}}\ \#]} = \frac{\prod_{p=3}^{pmax}(p-2)}{\prod_{p=2}^{pmax}p} = \frac{1}{2} * \prod_{p=3}^{pmax}\frac{(p-2)}{p}$

multiplying and dividing the second term of the same by (p -1) we obtain:

(4.2.4) $Dncncomp_{]0,\ \sqrt{2N_{0pm}}\#]} = \frac{1}{2} * \prod_{p=3}^{pmax} \frac{(p-2)}{p} * \frac{(p-1)}{(p-1)} = \frac{1}{2} * \prod_{p=3}^{pmax} \frac{(p-1)}{p} * \prod_{p=3}^{pmax} \frac{(p-2)}{(p-1)} = \prod_{p=2}^{pmax} \frac{(p-1)}{p} * \prod_{p=3}^{pmax} \frac{(p-2)}{(p-1)}$

In the last member of (4.2.4) we have substituted for $\frac{1}{2} * \prod_{p=3}^{pmax} \frac{(p-1)}{p}$ the term $\prod_{p=2}^{pmax} \frac{(p-1)}{p}$ which, as we know from (2.2.2), corresponds, for $N_0 \geq 121$, to the average density $Dnc_{]0,\sqrt{2N_{0pm}}\#]}$ of the numbers M existing in the interval $]0, \sqrt{(2N_{0pm})}\#]$ **not congruent with $N_{0pm}$ for only the p-modules$_i$** belonging to the set $\mathbb{P}\left(\sqrt{(2N_{0pm})}\right)$; in these last three formulae $p_{max}$ is obviously equal to the highest prime number less than the $\sqrt{(2N_{0pm})}$.

Let us then see if we can find a relationship between $\prod_{p=3}^{pmax} \frac{(p-2)}{(p-1)}$ e $\prod_{p=2}^{pmax} \frac{(p-1)}{p}$ so that we can determine the value of $Dncncomp_{]0,\ \sqrt{2N_{0pm}}\#]}$ as a function of $Dnc_{]0,\sqrt{2N_{0pm}}\#]}$.

We can write:

(4.2.5) $\frac{\prod_{p=3}^{pmax}\frac{(p-2)}{(p-1)}}{\prod_{p=2}^{pmax}\frac{(p-1)}{p}} = \prod_{p=3}^{pmax} \frac{(p-2)}{(p-1)} * \prod_{p=2}^{pmax} \frac{p}{(p-1)} = \prod_{p=3}^{pmax} \frac{(p-2)}{(p-1)} * 2 * \prod_{p=3}^{pmax} \frac{p}{(p-1)} = 2 * \prod_{p=3}^{pmax} \frac{p*(p-2)}{(p-1)^2}$

where one can easily verify that the relationship between the term $\prod_{p=3}^{pmax} \frac{(p-2)}{(p-1)}$ and that $\prod_{p=2}^{pmax} \frac{(p-1)}{p}$ for $N_{0pm}$ = 127 (the first "prime" following 121) takes on the value 0.6767578125, for $N_0$ =9006001 the value 0.6601862196 and then, as N increases$_0$ towards infinity, and thus extending the product over all prime numbers ≥ 3, tends rapidly to decrease towards the constant of the prime twins $C_2$ that appears in the Hardy-Littlewood conjecture [(c)] on the distribution of prime twins:

$\prod_{p\geq 3} \frac{p*(p-2)}{(p-1)^2} = C_2 \approx 0.66016118138468695739278121100 14$ ..........

We can therefore write:

(4.2.6) $\prod_{p=3}^{pmax} \frac{(p-2)}{(p-1)} \approx 2*C\ *_2 \prod_{p=2}^{pmax} \frac{(p-1)}{p}$

which substituted in (4.2.4) gives us:

$Dncncomp_{]0,\ \sqrt{2N_{0pm}}\#]} \approx \prod_{p=2}^{pmax} \frac{(p-1)}{p} * \prod_{p=3}^{pmax} \frac{(p-2)}{(p-1)} \approx 2*C\ *_2 (\prod_{p=2}^{pmax} \frac{(p-1)}{p})^2$

whence:

(4.2.7) $Dncncomp_{]0,\ \sqrt{2N_{0pm}}\#]} \approx 2*C_2 *(Dnc_{]0,\sqrt{2N_{0pm}}\#]})^2$

In this relation $Dncncomp_{]0,\ \sqrt{2N_{0pm}}\#]}$ to the square of $Dnc_{]0,\sqrt{2N_{0pm}}\#]}$ the constant $C_2$ changes little with the variation of $N_0$ when this is equal to a prime number $N_{0pm}$ . If we apply (4.2.7) to the interval $]0, N_{0pm}]$ it can be shown (Appendix B) that a negligible relative approximation of 0.1812 is made for $N_{0pm}$ = 127, and rapidly decreasing for higher values of $N_{0pm}$ (0.0404 for $N_{0pm}$ = 1277). An

approximation which, incidentally, for low values of $N_{0pm}$ is compensated for by the corresponding higher values of $C_2$ .
It can therefore be written with good approximation:

(4.2.8) $Dncncomp_{]0,\ N_0]}\ {}_2 * 2{*}C \approx (Dnc_{]0, N_{0pm}]})^2$

**Regardless of the distribution law of (4.2.8), the full analogy existing (with N0 prime) between (3.3.7) and (3.3.8) concerning prime twins and (4.2.7) and (4.2.8) leads us to believe that $Dncncomp_{]0,\ N_0]}$ as well as $Dncncomp(PG)_{]0,\ N_0]}$ is surely greater than 1 (there being in each interval ]0, $N_{0pm}$] with $N_{0pm} \geq 127$ well more than one pair of prime twins) and thus in line with Goldbach's conjecture.**

*Remark* **4.2.9** *In (4.2.8) as stated in Remarks (4.2.2) and (2.2.6) both Dncncomp(Ɠ)$_{(N0pm)}$ and $Dnc_{]0, N_{0pm}]}$) do not include the possible $n_0$ for which the ($N_{0pm}$ - $n_0$ ) are equal to the primes less than or equal to the $\sqrt{(2N_{0pm})}$ but since this relation is always valid $\forall N_{0pm} \in \mathbb{P}$ starting from $N_{0pm}$ = 127 (the first "prime" following 121) we can extend (4.2.8) to all prime numbers (except 2, 3, 5, 7, 11, 13) less than any $N_{0pm}$ greater than or equal to 127. In fact for $N_{0pm}$ = 127 (and thus $\sqrt{(2N_{0pm})}$ = $\sqrt{254}$ = 15.93, $p_{max}$ =13), (4.2.8) applies to all the $n_0$ prisotto and prisopra of $N_{0pm}$ which when subtracted from $N_{0pm}$ result in primes between 14 and 127; for $N_{0pm}$ = 131 (and therefore $\sqrt{262}$ = 16.18, $p_{max}$ =13), (4.2.8) concerns primes between 14 and 131; for $N_{0pm}$ = 137 (and therefore $\sqrt{274}$ = 16.55, $p_{max}$ =13), (4.2.8) still concerns primes between 14 and 137; for $N_{0pm}$ = 149 (and thus $\sqrt{298}$ = 17.26, $p_{max}$ =17), (4.2.8) still concerns primes between 18 and 149; and we can continue like this for all subsequent $N_{0pm}$ .*

But it can be verified, assuming $N_{0pm}$ =127 and thus $C_2$ =0.6767, that (*4.2.8*), with an approximation of about 14%, also holds if we take into account the primes 2, 3, 5, 7, 11, 13. In fact with $N_{0pm}$ =127 there are 31 primes and 9 numbers Ɠ whence (4.2.9) becomes:

$$\frac{9}{127} \approx 2{*}0{,}6767{*}\left(\frac{31}{127}\right)^2$$

0,07086 ≈ 0,08063

Obviously, as N increases$_{0pm}$ , subject to the validity of (4.2.8) for all primes greater than 13, the approximation decreases.
Ultimately $\forall N_{0pm} \in \mathbb{P}$ e greater than or equal to 127 we can still consider (*4.2.8*) to be valid even for primes less than $N_{0pm}$ and thus, substitute in it the density $D_{Ɠ(N0pm)}$ of the numbers Ɠ ≤ $N_{0pm}$ instead of $Dncncomp_{]0,\ N_0]}$ and the density $Dprimi_{]0, N_{0pm}]}$ of primes less than or equal to $N_{0pm}$ in place of $Dnc_{]0, N_{0pm}]}$).
Consequently, the relation follows from (4.2.8):

(4.2.10) $D_{Ɠ(N0pm)} \approx 2{*}C_2 {*}(Dprimi_{]0, N_{0pm}]})^2$

Being then for the NPT $Dprimi_{]0, N_{0pm}]} = \frac{1}{logN_0}$ one can write:

(4.2.11) $D_{Ɠ(N0pm)} \approx 2{*}C_2 {*}(\frac{1}{logN_{0pm}})^2$

To calculate the number $M_{Ǥ(N_{0pm})}$ of the numbers Ǥ smaller than $N_{0pm}$ multiply both members of (4.2.11) by $N_{0pm}$ :

(4.2.12) $M_{Ǥ(N_{0pm})} = D_{Ǥ(N_{0pm})} * N_{0pm} \approx N_{0pm}*2*C_2*(\frac{1}{logN_{0pm}})^2$

Appendix D gives an example of $n_0$ prisotto and prisopra of $N_{0pm}$ and the corresponding values of $Dnc_{]0,\, N_{0pm}]}$, $Dncncomp_{]0,\, N_0]]}$ and $M_G$ verified and calculated.

It is emphasised that the relation (4.2.12) bears a close resemblance to Vinogradov's theorem[1] .

**Remark 4.2.13** *Since the expression* $N*_{0pm}(\frac{1}{logN_{0pm}})^2$ *for* $N_{0pm} = 127$ *takes on a value approximately equal to 5, which increases for* $N_{0pm} > 12$ *and, the product* $2*C_2$ *is always greater than 1,* $M_{Ǥ(N_{0pm})}$ *will always be greater than or equal to 1. This is confirmed by the fact that with* $N_{0pm}$ *prime there will always be at least one number Ǥ = 0.*

It follows from (4.2.12) that for all prime numbers $N_{0pm}$ the numbers equal to their doubles ($2*N_{0pm}$ ) are always the sum of one or more pairs of primes.

For $N_0$ other than $N_{0pm}$ the $Dncncomp_{]0,\ \sqrt{2N_0}\#]}$ (4.2.3) is modified in the expression:

(4.2.14) $Dncncomp_{]0,\ \sqrt{2N_0}\#]} = \frac{1}{2} * \prod_{3 \le p_l \le p_{max}} \frac{(p_l - 2)}{p_l} * \prod_{3 \le p_j \le p_{max}} \frac{(p_j - 1)}{p_j}$

in which the first $p_j$ belonging to $\mathbb{P}(\sqrt{(2N_0)})$ appear distinct in the $p_j$ equal to the factors of $N_0$ and in those $p_l$ that are not (see section 4.2 (b) and (c)). But (4.2.14) can also be written like this:

(4.2.15) $Dncncomp_{]0,\ \sqrt{2N_0}\#]} = \frac{1}{2} * \prod_{3 \le p_i \le p_{max}} \frac{(p_i - 2)}{p_i} * \prod_{3 \le p_j \le p_{max}} \frac{(p_j - 1)}{(p_j - 2)}$

Knowing that the value of $p_{\max}$ of (4.2.3) and (4.2.15) remains the same for each interval $]0, N_0]$ with $N_0$ such that it results $p_{\max} < \sqrt{2N_0} < p_{\max succ}$ where $p_{\max}$ is the highest prime less than $\sqrt{2N_{0pm}}$ e $p_{\max succ}$ the first immediately following $p_{\max}$, by comparing (4.2.15), where

[1] Vinogradov's theorem [(d)] states that any sufficiently large odd integer can be written as the sum of c primes with c ≥ 3 . The above theorem is only proved for c ≥ 3, whereas for c = 2 it becomes a (Goldbach's extended) conjecture and the number of pairs of equal primes whose sum equals an even n is expressed by the following relation:

$$2\Pi_2 \left( \prod_{\substack{p|n \\ p\ge 3}} \frac{p-1}{p-2} \right) \int_2^n \frac{dx}{\ln^2 x} \approx 2\Pi_2 \left( \prod_{\substack{p|n \\ p\ge 3}} \frac{p-1}{p-2} \right) \frac{n}{\ln^2 n},$$

Where the term $\Pi_2$ is the constant of the prime twins. If we substitute the term 2M at even n, with M being prime, the term

$$\left( \prod_{\substack{p|n \\ p\ge 3}} \frac{p-1}{p-2} \right)$$

fails since n=2M is not divisible by any prime ≥ 3 and Vinogradov's formula with c=2 becomes the same as our (4.2.10) proved for any M ≥ 127

$Dncncomp_{]0,\ p_{max}\#]}$ is relative to any $N_0$ other than Nopm, and (4.2.3) relative to the first highest Nopm < No results:

(4.2.16) $Dncncomp_{]0,\ \sqrt{2N_0}\#]} = Dncncomp_{]0,\ \sqrt{2N_{0pm}}\#]} * \prod_{3 \le p_j \le p_{max}} \frac{(p_j-1)}{(p_j-2)}$

where both densities refer to the same interval $]0, p_{max}\#]$ with p # = $_{max}$ $\sqrt{2N_0}$ # = $\sqrt{2N_{0pm}}$ # but are of integers of the interval incongruous and incompcongruous with two different numbers: $N_0$ and $N_{0pm}$

According to (4.2.7), (4.2.16) becomes:

(4.2.17) $Dncncomp_{]0,\ \sqrt{2N_0}\#]} = 2*C_2 *(Dnc_{]0,\ \sqrt{2N_{0pm}}\#]})^{*\,2} \prod_{3 \le p_j \le p_{max}} \frac{(p_j-1)}{(p_j-2)} =$

$= 2*C_2 *(Dnc_{]0,\ \sqrt{2N_0}\#]})^{*\,2} \prod_{3 \le p_j \le p_{max}} \frac{(p_j-1)}{(p_j-2)}$

being by hypothesis ($p_{max} < \sqrt{2N_0} < p_{max\,succ}$ where $p_{max}$ is the highest prime less than $\sqrt{2N_{0pm}}$) $Dnc_{]0,\ \sqrt{2N_{0pm}}\#]} = Dnc_{]0,\ \sqrt{2N_0}\#]}$.

Finally, since the term $\prod_{3 \le p_j \le p_{max}} \frac{(p_j-1)}{(p_j-2)} > 1$, (4.2.17) in turn becomes:

(4.2.18) $Dncncomp_{]0,\ \sqrt{2N_0}\#]} > 2*C_2 *(Dnc_{]0,\ \sqrt{2N_0}\#]})^2$

The inequality relation (4.2.18), similarly to the equality relation (4.2.7), following the same reasoning as in Appendix B, has a negligible relative approximation when referred to the interval ]0, N ].$_0$

This allows us to apply (4.2.18) to the interval $]0, N]_0$ in addition to the interval $]0, \sqrt{2N_{0pm}}\#]$ allowing us to write::

(4.2.19) $Dncncomp_{]0, N_0]}\ 2 * 2*C \ >(Dnc_{]0, N_0]})^2$

However, on the basis of observation (4.2.9) we can assume that ∀ $N_0$ ∈ N greater than 121, (4.2.19) remains valid for all primes smaller than $N_0$ and so we can substitute the density $D(Ꞡ)_{(N0)}$ of all numbers Ꞡ smaller than $N_0$ in place of $Dncncomp_{]0, N_0]}$ e $Dprimi_{]0, N_0]}$ al posto di $Dnc_{]0, N_0]}$, by writing:

(4.2.20) $D(Ꞡ)_{(N0)} > 2*C_2 *(Dprimi_{]0, N_0]})^2$

If we now apply the NPT and multiply both members of (4.2.20) by $N_0$ we obtain the number $MꞠ_{(N0)}$ of the numbers Ꞡ smaller than $N_0$ :

(4.2.21) $MꞠ_{(N0)} = D(Ꞡ)_{(N0)} *N_0 > N_0*2*C_2 *(\frac{1}{logN_0})^2$

where $N *_0(\frac{1}{logN_0})^2$ always takes a value greater than or equal to 1 for $N_0 \ge 2$. Consequently, since $2*C_2$ is also always greater than 1, $MꞠ_{(N0)}$ will always be greater than or equal to 1.

It therefore follows from (4.2.21) that even for all numbers $N_0 \neq N_{0pm}$ the numbers Ꞡ are always greater than or equal to 1 and thus there will always be at least one pair of primes ($N_0$ - Ꞡ and $N_0$ + Ꞡ ) whose sum is equal to $2*N_0$ as predicted by Goldbach's conjecture.
For $N_0$ less than 121, Goldbach's conjecture is easily verifiable.

*APPENDIX A*

**Table showing the bi-univocal correspondence between the numbers 1 to 210 and all combinations of the congruence classes modulo 2 - 3 - 5- 7**

| **no. of modules** | **no. of modules** | **no. of modules** |
|---|---|---|
| 2 - 3 - 5 - 7 | 2 - 3 - 5 - 7 | 2 - 3 - 5 - 7 |
| 1) 1 - 1 - 1 - 1 | 71) 1 - 2 - 1 - 1 | 141) 1 - 0 - 1 - 1 |
| 2) 0 - 2 - 2 - 2 | 72) 0 - 0 - 2 - 2 | 142) 0 - 1 - 2 - 2 |
| 3) 1 - 0 - 3 - 3 4 | 73) 1 - 1 - 3 - 3 | 143) 1 - 2 - 3 - 3 |
| ) 0 - 1 - 4 - 4 | 74) 0 - 2 - 4 - 4 | 144) 0 - 0 - 4 - 4 |
| 5) 1 - 2 - 0 - 5 | 75) 1 - 0 - 0 - 5 | 145) 1 - 1 - 0 - 5 |
| 6) 0 - 0 - 1 - 6 | 76) 0 - 1 - 1 - 6 | 146) 0 - 2 - 1 - 6 |
| 7) 1 - 1 - 2 - 0 | 77) 1 - 2 - 2 - 0 | 147) 1 - 0 - 2 - 0 |
| 8) 0 - 2 - 3 - 1 | 78) 0 - 0 - 3 - 1 | 148) 0 - 1 - 3 - 1 |
| 9) 1 - 0 - 4 - 2 | 79) 1 - 1 - 4 - 2 | 149) 1 - 2 - 4 - 2 |
| 10) 0 - 1 - 0 - 3 | 80) 0 - 2 - 0 - 3 | 150) 0 - 0 - 0 - 3 |
| 11) 1 - 2 - 1 - 4 | 81) 1 - 0 - 1 - 4 | 151) 1 - 1 - 1 - 4 |
| 12) 0 - 0 - 2 - 5 | 82) 0 - 1 - 2 - 5 | 152) 0 - 2 - 2 - 5 |
| 13) 1 - 1 - 3 - 6 | 83) 1 - 2 - 3 - 6 | 153) 1 - 0 - 3 - 6 |
| 14) 0 - 2 - 4 - 0 | 84) 0 - 0 - 4 - 0 | 154) 0 - 1 - 4 - 0 |
| 15) 1 - 0 - 0 - 1 | 85) 1 - 1 - 0 - 1 | 155) 1 - 2 - 0 - 1 |
| 16) 0 - 1 - 1 - 2 | 86) 0 - 2 - 1 - 2 | 156) 0 - 0 - 1 - 2 |
| 17) 1 - 2 - 2 - 3 | 87) 1 - 0 - 2 - 3 | 157) 1 - 1 - 2 - 3 |
| 18) 0 - 0 - 3 - 4 | 88) 0 - 1 - 3 - 4 | 158) 0 - 2 - 3 - 4 |
| 19) 1 - 1 - 4 - 5 | 89) 1 - 2 - 4 - 5 | 159) 1 - 0 - 4 - 5 |
| 20) 0 - 2 - 0 - 6 | 90) 0 - 0 - 0 - 6 | 160) 0 - 1 - 0 - 6 |
| 21) 1 - 0 - 1 - 0 | 91) 1 - 1 - 1 - 0 | 161) 1 - 2 - 1 - 0 |
| 22) 0 - 1 - 2 - 1 | 92) 0 - 2 - 2 - 1 | 162) 0 - 0 - 2 - 1 |
| 23) 1 - 2 - 3 - 2 | 93) 1 - 0 - 3 - 2 | 163) 1 - 1 - 3 - 2 |
| 24) 0 - 0 - 4 - 3 | 94) 0 - 1 - 4 - 3 | 164) 0 - 2 - 4 - 3 |
| 25) 1 - 1 - 0 - 4 | 95) 1 - 2 - 0 - 4 | 165) 1 - 0 - 0 - 4 |
| 26) 0 - 2 - 1 - 5 | 96) 0 - 0 - 1 - 5 | 166) 0 - 1 - 1 - 5 |
| 27) 1 - 0 - 2 - 6 | 97) 1 - 1 - 2 - 6 | 167) 1 - 2 - 2 - 6 |
| 28) 0 - 1 - 3 - 0 | 98) 0 - 2 - 3 - 0 | 168) 0 - 0 - 3 - 0 |
| 29) 1 - 2 - 4 - 1 | 99) 1 - 0 - 4 - 1 | 169) 1 - 1 - 4 - 1 |
| 30) 0 - 0 - 0 - 2 | 100) 0 - 1 - 0 - 2 | 170) 0 - 2 - 0 - 2 |
| 31) 1 - 1 - 1 - 3 | 101) 1 - 2 - 1 - 3 | 171) 1 - 0 - 1 - 3 |
| 32) 0 - 2 - 2 - 4 | 102) 0 - 0 - 2 - 4 | 172) 0 - 1 - 2 - 4 |
| 33) 1 - 0 - 3 - 5 | 103) 1 - 1 - 3 - 5 | 173) 1 - 2 - 3 - 5 |
| 34) 0 - 1 - 4 - 6 | 104) 0 - 2 - 4 - 6 | 174) 0 - 0 - 4 - 6 |
| 35) 1 - 2 - 0 - 0 | 105) 1 - 0 - 0 - 0 | 175) 1 - 1 - 0 - 0 |
| 36) 0 - 0 - 1 - 1 | 106) 0 - 1 - 1 - 1 | 176) 0 - 2 - 1 - 1 |
| 37) 1 - 1 - 2 - 2 | 107) 1 - 2 - 2 - 2 | 177) 1 - 0 - 2 - 2 |
| 38) 0 - 2 - 3 - 3 | 108) 0 - 0 - 3 - 3 | 178) 0 - 1 - 3 - 3 |
| 39) 1 - 0 - 4 - 4 | 109) 1 - 1 - 4 - 4 | 179) 1 - 2 - 4 - 4 |
| 40) 0 - 1 - 0 - 5 | 110) 0 - 2 - 0 - 5 | 180) 0 - 0 - 0 - 5 |
| 41) 1 - 2 - 1 - 6 | 111) 1 - 0 - 1 - 6 | 181) 1 - 1 - 1 - 6 |
| 42) 0 - 0 - 2 - 0 | 112) 0 - 1 - 2 - 0 | 182) 0 - 2 - 2 - 0 |
| 43) 1 - 1 - 3 - 1 | 113) 1 - 2 - 3 - 1 | 183) 1 - 0 - 3 - 1 |
| 44) 0 - 2 - 4 - 2 | 114) 0 - 0 - 4 - 2 | 184) 0 - 1 - 4 - 2 |
| 45) 1 - 0 - 0 - 3 | 115) 1 - 1 - 0 - 3 | 185) 1 - 2 - 0 - 3 |
| 46) 0 - 1 - 1 - 4 | 116) 0 - 2 - 1 - 4 | 186) 0 - 0 - 1 - 4 |
| 47) 1 - 2 - 2 - 5 | 117) 1 - 0 - 2 - 5 | 187) 1 - 1 - 2 - 5 |
| 48) 0 - 0 - 3 - 6 | 118) 0 - 1 - 3 - 6 | 188) 0 - 2 - 3 - 6 |
| 49) 1 - 1 - 4 - 0 | 119) 1 - 2 - 4 - 0 | 189) 1 - 0 - 4 - 0 |
| 50) 0 - 2 - 0 - 1 | 120) 0 - 0 - 0 - 1 | 190) 0 - 1 - 0 - 1 |
| 51) 1 - 0 - 1 - 2 | 121) 1 - 1 - 1 - 2 | 191) 1 - 2 - 1 - 2 |
| 52) 0 - 1 - 2 - 3 | 122) 0 - 2 - 2 - 3 | 192) 0 - 0 - 2 - 3 |
| 53) 1 - 2 - 3 - 4 | 123) 1 - 0 - 3 - 4 | 193) 1 - 1 - 3 - 4 |
| 54) 0 - 0 - 4 - 5 | 124) 0 - 1 - 4 - 5 | 194) 0 - 2 - 4 - 5 |
| 55) 1 - 1 - 0 - 6 | 125) 1 - 2 - 0 - 6 | 195) 1 - 0 - 0 - 6 |
| 56) 0 - 2 - 1 - 0 | 126) 0 - 0 - 1 - 0 | 196) 0 - 1 - 1 - 0 |
| 57) 1 - 0 - 2 - 1 | 127) 1 - 1 - 2 - 1 | 197) 1 - 2 - 2 - 1 |
| 58) 0 - 1 - 3 - 2 | 128) 0 - 2 - 3 - 2 | 198) 0 - 0 - 3 - 2 |
| 59) 1 - 2 - 4 - 3 | 129) 1 - 0 - 4 - 3 | 199) 1 - 1 - 4 - 3 |
| 60) 0 - 0 - 0 - 4 | 130) 0 - 1 - 0 - 4 | 200) 0 - 2 - 0 - 4 |
| 61) 1 - 1 - 1 - 5 | 131) 1 - 2 - 1 - 5 | 201) 1 - 0 - 1 - 5 |
| 62) 0 - 2 - 2 - 6 | 132) 0 - 0 - 2 - 6 | 202) 0 - 1 - 2 - 6 |
| 63) 1 - 0 - 3 - 0 | 133) 1 - 1 - 3 - 0 | 203) 1 - 2 - 3 - 0 |
| 64) 0 - 1 - 4 - 1 | 134) 0 - 2 - 4 - 1 | 204) 0 - 0 - 4 - 1 |
| 65) 1 - 2 - 0 - 2 | 135) 1 - 0 - 0 - 2 | 205) 1 - 1 - 0 - 2 |
| 66) 0 - 0 - 1 - 3 | 136) 0 - 1 - 1 - 3 | 206) 0 - 2 - 1 - 3 |
| 67) 1 - 1 - 2 - 4 | 137) 1 - 2 - 2 - 4 | 207) 1 - 0 - 2 - 4 |
| 68) 0 - 2 - 3 - 5 | 138) 0 - 0 - 3 - 5 | 208) 0 - 1 - 3 - 5 |
| 69) 1 - 0 - 4 - 6 | 139) 1 - 1 - 4 - 6 | 209) 1 - 2 - 4 - 6 |
| 70) 0 - 1 - 0 - 0 | 140) 0 - 2 - 0 - 0 | 210) 0 - 0 - 0 - 0 |

**Preamble** If we wish to apply (4.2.7) to the interval ]0, $N_{0pm}$ ] as well as to the interval ]0, $p_{max}$ #], with $p_{max}$ equal to the first higher less than or equal to the $\sqrt{2N_{0pm}}$, we must take into account that while in the interval ]0, $p_{max}$ #] both $Dnc_{]0,\sqrt{2N_{0pm}}\#]}$ and $Dncncomp_{]0,\sqrt{2N_{0pm}}\#]}$ are equal to the product of the factors (p - x)/p [where x is equal to 1 for $Dnc_{]0,\sqrt{2N_{0pm}}\#]}$ and 2 for $Dncncomp_{]0,\sqrt{2N_{0pm}}\#]}$ ] and where p varies respectively between 2 and $p_{max}$ and between 3 and $p_{max}$ , on the other hand in the interval ]0, $N_{0pm}$ ], the two densities $Dnc_{]0,\, N_{0pm}\,]}$ e $Dncncomp_{]0,\, N_{0pm}\,]}$ are no longer equal to the product of the factors (p - x)/p since $N_{0pm}$ unlike $p_{max}$ # is not a multiple of any prime in the interval [2, $p_{max}$ ]. It can be shown, however, that the ratio of the density $Dncncomp_{]0,\, N_{0pm}\,]}$ and the square of the density $Dnc_{]0,\, N_{0pm}\,]}$ relative to the interval ]0, $N_{0pm}$ ] is almost equal to that of (4.2.7) with a relative approximation of 0.1812 for $N_{0pm}$ =127 and rapidly decreasing for higher values (0.0404 for $N_{0pm}$ =1277).

To this end, we compute the individual densities for a single p_1 modulus Ɛ {2, 3, 5......., $p_{max}$ } of the integers in the interval ]0, $N_{0pm}$ ] not congruent to $N_{0pm}$ modulus p_1 (for x=1) and not congruent to $N_{0pm}$ modulus p_1 (for x=2). Since these densities in the interval ]0, $N_{0pm}$ ] are different from (p_1 - x) / p_1, in order to calculate them, for each p_1 we divide the interval ]0, $N_{0pm}$ ] into two intervals ]0, $X_{p\_1}$ ] and ]$X_{p\_1}$ , $N_{0pm}$ ] where $X_{p\_1}$ is the maximum multiple of p_1 contained in the interval ]0, $N_{0pm}$ ]. We then compute the total densities of these numbers (p_1 - x), non-congruent to $N_{0pm}$ modulo p_1 (for x=1) and non-congruent to $N_{0pm}$ modulo p_1 (for x=2), present in the two intervals:

$$a)\ D(p\_1) = \frac{X_{p\_1} * \frac{(p\_1 - x)}{p\_1} + ([N_{0pm}]_{p\_1} - f(h,x))}{N_{0pm}} = \frac{L * (p_1 - x) + ([N_{0pm}]_{p_1} - f(h,x))}{N_{0pm}}$$

where L is equal to the ratio of the maximum multiple $X_{p\_1}$ of p_1 contained in the interval ]0, $N_{0pm}$ ] to p_1, where $[N\,]_{0pmp\_1} < p\_1$ is equal to the width of the interval ] $X_{p\_1}$ , $N_{0pm}$ ], where h is the number of integers in the interval ] $X_{p\_1}$ , $N_{0pm}$ ] whose moduli p_1, in the case of x=1, is equal to the remainder of the division of $N_{0pm}$ by p_1 and, in the case of x=2, are equal to the remainder or its complement of the division of $N_{0pm}$ by p_1, and where finally f(h,x) is a function of h and x that takes the values of 1 or 2 depending on the values of h and x. Since $N_{0pm}$ is obviously a congruent number with itself h will be equal to 1 or 2.

Knowing that [$N_{0pm}$ ]p_1 can take on a value between 1 and p_1 - 1 (the value 0 being excluded since both $N_{0pm}$ and p_1 are primes) and that f(h,x), depending on the value of h and x, can be worth 1 or 2, we can state that the term ($[N\,]_{0pmp\_1}$ - f(h,x)), always choosing for the purpose of our demonstration the highest of the possible values, takes on a value according to the following scheme:

**if h=1 for both x=1 and x=2 we have**: ($[N\,]_{0pmp\_1}$ - f(h,x)) = $[N\,]_{pmp\_1}$ - 1
**if h=2 and therefore x=2 we have**: ($[N\,]_{0pmp\_1}$ - f(h,x)) = $[N\,]_{0pmp\_1}$ - 2

We can therefore write:

$$b)\ L = \frac{(N_{0pm} - [N_0 pm]_{p\_1})}{p\_1}$$

and (a):

c) $D(p\_1) = \left\{\frac{\left(N_{0pm} - [N_{0pm}]_{p\_1}\right)}{p\_1} * (p\_1 - x) + \left([N_{0pm}]_{p\_1} - f(h,x)\right)\right\} * \frac{1}{N_{0pm}}$

and then multiplying and dividing the term $([N]_{0pmp\_1} - f(h,x))$ by p_1:

d) $D(p\_1) = \frac{N_{0pm}*(p\_1-x) - [N_{0pm}]_{p\_1}*p\_1 + [N_{0pm}]_{p\_1}*x + ([N_0pm]_{p\_1} - f(h,x))*p\_1}{N_{0pm}*p\_1}$

e) $D(p\_1) = \frac{N_{0pm}*(p\_1-x) - [N_{0pm}]_{p\_1}*p\_1 + [N_{0pm}]_{p\_1}*x + [N_{0pm}]_{p\_1}*p\_1 - f(h,x)*p\_1}{N_{0pm}*p\_1}$

f) $D(p\_1) = \frac{N_{0pm}*(p\_1-x) + [N_{0pm}]_{p\_1}*x - f(h,x)*p\_1}{N_{0pm}*p\_1}$

g) $D(p\_1) = 1 - \frac{N_{0pm}*x}{N_{0pm}*p\_1} - \frac{f(h,x)*p_1 - [N_{0pm}]_{p\_1}*x}{N_{0pm}*p\_1}$

Let us now see, based on the possible values of h (1, 2) and x (1, 2) what expression g) takes on.

For h=1 and x=1, 2 and thus f(h,x)=1 we have that:

h) $D(p\_1) = 1 - \frac{x}{p\_1} - \frac{p_1 - [N_{0pm}]_{p\_1}*x}{N_{0pm}*p\_1}$

For h=2 and x=2 and thus f(h,x)=2 we have that:

i) $D(p\_1) = 1 - \frac{x}{p\_1} - \frac{2p\_1 - [N_{0pm}]_{p\_1}*x}{N_{0pm}*p\_1}$

**Definition** We define $Dh_{N0pm}$ the product of the respective individual densities denoted by h) (with x=1) for each of the p_1 less than or equal to $p_{max}$ and by $Di_{N0pm}$ the product of the respective individual densities denoted by i) (with x=2) for each of the p_1 less than or equal to $p_{max}$ (with $p_{max}$ equal to the first highest less than or equal to the $\sqrt{2N_{0pm}}$).

**Lemma (a)** *The relative approximation $a_{rt}$ between the ratio $Di_{N0pm}/Dh_{N0pm}{}^2$ and that $(Dnncompc_{]0,\sqrt{2N_{0pm}}\#]})/\left(Dnc_{]0,\sqrt{2N_{0pm}}\#]}\right)^2$ is equal to:*

$t)\ a_{rt} = \frac{4*\sqrt{2}}{\sqrt{\text{Nopm}}*ln\sqrt{2\text{Nopm}}}$

We begin by calculating the approximation of the $Dh_{N0pm}$ with respect to the $Dnc_{]0,\sqrt{2N_{0pm}}\#]}$ and of $Di_{N0pm}$ with respect to the $Dncncomp_{]0,\sqrt{2N_{0pm}}\#]}$. In the expressions (h) and (i) the first two terms of the second member represent the individual densities (p_1 -1)/p_1 that the n € ]0, $p_{max}$ #] are either non-congruent with $N_{0pm}$ modulo p_1 (with x=1) or non-congruent with $N_{0pm}$ modulo p_1 (with x=2) in the interval ]0, $p_{max}$ #] . The last term of the second member then represents the approximation that the single density D(p_1) has to the single density (p_1 -1)/p_1 in the interval ]0, $N_{0pm}$ ]. For the purposes of our demonstration, we must consider in expressions h) and i) among the various possible values of $[N_{0pm}]_{p\_1}$ the one with the largest approximation in order to verify that it does not compromise the final result of the demonstration. Let us then refer to a single expression of D(p_1) and calculate its relative approximation:

j) $D(p\_1) = \frac{p\_1 - x}{p\_1} - \frac{x * p_1 - [N_{0pm}]_{p\_1} * x}{N_{0pm} * p\_1}$

with relative approximation $a_r = \frac{x * p\_1 - [N_{0pm}]_{p\_1} * x}{N_{0pm} * p\_1} * \frac{p\_1}{p\_1 - x}$

(j) **for x=1** becomes:

k) $D_1(p\_1) = 1 - \frac{1}{p\_1} - \frac{p\_1 - [N_{0pm}]_{p\_1}}{N_{0pm} * p\_1}$

which, assuming $[N_{0pm}]_{p\_1} = 1$ again to choose the maximum approximation, becomes:

l) $D_1(p\_1) = 1 - \frac{1}{p\_1} - \frac{1}{N_{0pm}} * \frac{p_{p\_1} - 1}{p\_1} = \frac{p\_1 - 1}{p\_1} - \frac{1}{N_{0pm}} * \frac{p_{p\_1} - 1}{p\_1}$

with relative approximation $a_r = \frac{1}{N_{0pm}} * \frac{p_{p\_1} - 1}{p\_1} * \frac{p\_1}{p\_1 - 1} = \frac{1}{N_{0pm}}$

and instead **for x=2** it becomes:

m) $D_2(p\_1) = \frac{p\_1 - 2}{p\_1} - \frac{2}{N_{0pm}} * \frac{p\_1 - 2}{p\_1}$

with relative approximation $a_r = \frac{2}{N_{0pm}} * \frac{p\_1 - 2}{p\_1} * \frac{p\_1}{p\_1 - 2} = \frac{2}{N_{0pm}}$

We have already defined products $Dh_{N0pm}$ e $Di_{N0pm}$ as:

n) $Dh_{N0pm} := \prod_{pi=2}^{p_{max}} D_1(p_i)$ e $Di_{N0pm} := \frac{1}{2} * \prod_{pi=3}^{p_{max}} D_2(p_i)$

from which it follows that the relationship between $Di_{N0pm}$ and the square of $Dh_{N0pm}$ is equal to:

o) $\frac{Di_{N0pm}}{Dh_{N0pm}^{2}} = \frac{\frac{1}{2} * \prod_{p=3}^{pmax} D_2(p_i)}{\left(\prod_{p=2}^{pmax} D_1(p_i)\right)^2}$

As in the case of expressions h) and i), also in the case of expressions l) and m) the first terms of the second member represent the densities (p_1 -1)/ p_1 and (p_1 -2)/ p_1 that the generic number n € ]0, $p_{max}$ #] is either non-congruent with $N_{0pm}$ modulo p_1 (with x=1) or non-congruent with $N_{0pm}$ modulo p_1 (with x=2). The last terms of the second member of the expressions l) and m) then represent the approximations that the densities $D_1$ (p_1) and $D_2$ (p_1) in the interval ]0, $N_{0pm}$ ] have with respect to those (p_1 -1)/ p_1 and (p_1 -2)/ p_1 relative to the interval ]0, $p_{max}$ #]).

Taking the expressions (2.2.2) and (4.2.3) from the text relating to the densities of the integers in the interval $]0, p_{max}\#]$ respectively incongruous and incompcongruous with $N_{0pm}$ modules $p_i \leq p_{max}$ :

$$p)\ Dnc_{]0,\sqrt{2N_{0pm}}\#]} = \prod_{p=2}^{p_{max}} \frac{(p-1)}{p} \qquad \text{and } q)\ Dncncomp_{]0,\sqrt{2N_{0pm}}\#]} = \frac{1}{2} * \prod_{p=3}^{pmax} \frac{(p-2)}{p}$$

we see that they are respectively the product of the first terms of $D_1$ (p_1) and $D_2$ (p_1) for as many p_1 ($p_i$) less than or equal to $p_{max}$ for x=1 and x=2 respectively. The relative approximation of the expressions n) to those p) and q), similarly to the error propagation, will be equal to the sum of the relative approximations of the individual terms $(p_i - x)/p_i$.

This means that for the expressions n) and o) there is a relative approximation $a_r$ equal to $\frac{x}{N_{0pm}} * \sum_{p_i \leq p_{max}} {p_i}^0$ which, for the NPT and knowing that $p_{max}$ is the highest prime less than or equal to the $\sqrt{2N_{0pm}}$, becomes:

$$r)\ a_r = \frac{x}{\text{Nopm}} * \frac{\sqrt{2\text{Nopm}}}{ln\sqrt{2\text{Nopm}}} = \frac{x * \sqrt{2}}{\sqrt{\text{Nopm}} * ln\sqrt{2\text{Nopm}}}$$

Relative approximation therefore that for $N_{0pm}$ = 127 is equal to: 0.0453*x while for $N_{0pm}$ = 1277 is equal to: 0.0101*x and continues to decrease for increasing values of $N_{0pm}$ .

From the expressions (p) and (q), it was derived in the text that the ratio between the density of the integers present in the interval $]0, p_{max}\#]$ incongruous and incompcongruous with $N_{0pm}$ modules $p_i \leq p_{max}$ and the square of the density of the integers present in the interval $]0, p_{max}\#]$ incongruous with $N_{0pm}$ modules $p_i \leq p_{max}$ is equal to

$$s)\ \frac{Dncncomp_{]0,\sqrt{2N_{0pm}}\#]}}{{Dnc_{]0,\sqrt{2N_{0pm}}\#]}}^2} = \frac{\frac{1}{2} * \prod_{p=3}^{pmax} \frac{(p-2)}{p}}{\left(\prod_{p=2}^{p_{max}} \frac{(p-1)}{p}\right)^2}$$

while the expressions n) showed that the ratio o) in the interval $]0, N_{0pm}]$ with respect to the ratio s) presents a relative approximation equal to the sum of the relative approximations of the terms present at the numerator and denominator; the term at the denominator presents an approximation double that indicated in r) and relative, with x=1, to the term $\prod_{p=2}^{p_{max}} \frac{(p-1)}{p}$ while the term at the numerator has an approximation indicated in r) and relative, with x=2, to the term $\frac{1}{2} * \prod_{p=3}^{pmax} \frac{(p-2)}{p}$.
Thus, the relative approximation of the ratio o) to that s), when instead of considering the interval $]0, p_{max}\ \#]$ one considers that $]0, N_{0pm}\ ]$, is equal to:

$$t)\ a_{rt} = \frac{2 * \sqrt{2}}{\sqrt{\text{Nopm}} * ln\sqrt{2\text{Nopm}}} + 2 * \frac{\sqrt{2}}{\sqrt{\text{Nopm}} * ln\sqrt{2\text{Nopm}}} = \frac{4 * \sqrt{2}}{\sqrt{\text{Nopm}} * ln\sqrt{2\text{Nopm}}}$$

so for $N_{0pm}$ = 127 $a_{rt}$ = 0.1812, for $N_0$ = 1277 $a_{rt}$ = 0.0404 and for $N_0$ increasing $a_{rt}$ continues to decrease.

So for the purposes of the demonstration, it can be written that the ratio of (4.2.8) in the text, in the worst case (i.e. to the highest approximation) is:

$$u)\ \frac{Di_{N0pm}}{{Dh_{N0pm}}^2} = \frac{Dncncomp_{]0,\sqrt{2N_{0pm}}\#]}}{(Dnc_{]0,\sqrt{2N_{0pm}}\#]})^2} * (1 - a_{rt})$$

$$v)\ \frac{Di_{N0pm}}{{Dh_{N0pm}}^2} \approx 2 * \text{C2} * (1 - a_{rt}) \approx 1{,}32 * (1 - a_{rt})$$

and consequently for $N_0 = 127$ ed $a_{rt} = 0.1812$ can be written:

$$w)\ \frac{Di_{N0pm}}{{Dh_{N0pm}}^2} \approx 2 * \text{C2} * (1 - a_{rt}) \approx 1{,}32 * 0{,}8188 \approx 1{,}0808$$

and for $N_0 = 1277$ ed $a_{rt} = 0{,}0404$:

$$x)\ \frac{Di_{N0pm}}{{Dh_{N0pm}}^2} \approx 2 * \text{C2} * (1 - a_{rt}) \approx 1{,}32 * 0{,}9596 \approx 1{,}2666$$

and for $N_0$ increasing, the ratio tends to 1.32.

Now in the interval ]0, $N_{0pm}$ ] analogous to the calculation of the $Dnc_{]0,\,N_{0pm}\,]}$, the product $Dh_{N0pm}$ of the respective individual densities denoted by h) (with x=1) for each p_1 less than or equal to $p_{max}$ approximates (Appendix E) without equating the correct density $Dnc_{]0,\,N_{0pm}\,]}$ of the uncongruent integers with $N_{0pm}$ modulo p_1 for each p_1 Ɛ {2, 3, 5......., $p_{max}$ } (with $p_{max}$ highest prime less than or equal to $\sqrt{2N_{0pm}}$ ). **In the following lemmas we will prove that $Dnc_{]0,\,N_{0pm}\,]}$ e $Dh_{N0pm}$ are almost asymptotically equivalent with a small relative error (equal to** $2{*}e^{-\gamma}$**) so that it is all internal to the approximation " ≈ " of (4.2.8).**

Similarly in the same interval ]0, $N_{0pm}$ ], $N_{0pm}$ being prime and therefore not a multiple of any number in the above interval, the product $Di_{N0pm}$ of the respective individual densities indicated by (i) (with x=2) for each of the p_1 less than or equal to $p_{max}$ approximates (Appendix E) without equalling the density $Dncncomp_{]0,\,N_{0pm}\,]}$ of the uncongruent integers with $N_{0pm}$ modulo p_1 for each p_1 Ɛ {2, 3, 5......., $p_{max}$ } (with $p_{max}$ highest prime less than or equal to $\sqrt{2N_{0pm}}$ ). And **in fact, in the following lemmas we shall prove that $Di_{N0pm}$ is almost asymptotically equivalent to $Dncncomp_{]0,\,N_{0pm}\,]}$.**

**Ultimately then we can write that:**

$$y)\ \frac{Dncncomp_{]0,\,N_{0pm}\,]}}{{Dnc_{]0,\,N_{0pm}\,]}}^2} \approx \frac{Di_{N0pm}}{{Dh_{N0pm}}^2} \approx 2 * \text{C2} * (1 - a_{rt}) \approx 1{,}32 * (1 - a_{rt})$$

**Observation** (x) implies that the ratio $\frac{Di_{N0pm}}{{Dh_{N0pm}}^2}$ tends to a constant, thus $Di_{N0pm}$ e ${Dh_{N0pm}}^2$ are almost asymptotically equivalent, which will also be demonstrated in a different way in lemma c).

**Lemma (b)** *The product $Dh_{N0pm}$ and the density $Dnc_{]0,\,N_{0pm}\,]}$ of integers in the interval ]0, Nopm] incongruous with Nopm are asymptotically nearly equivalent functions (with a relative error of $2{*}e^{-\gamma}$ ).*

First of all, we can say that according to the NPT the density $Dnc_{]0,\,N_{0pm}\,]}$ of incongruous numbers with $N_{0pm}$ modulo p_1 for each p_1 Ɛ {2, 3, 5......., $p_{max}$ } (with $p_{max}$ highest prime less than or equal to the $\sqrt{2N_{0pm}}$ ) in the interval ]0, $N_{0pm}$ ] is given, see also (2.2.5) in the text, by the difference between the primes contained in the interval ]0, $N_{0pm}$ ] and those contained in the interval ]0, $\sqrt{2N_{0pm}}$] divided by $N_{0pm}$ and i.e. by:

$$Dnc_{]0,\,N_{0pm}\,]} = \frac{\left(\frac{N_{0pm}}{\log N0pm} - \frac{\sqrt{2N_{0pm}}}{\log\sqrt{2N_{0pm}}}\right)}{Nopm} \approx \frac{1}{\log N_{opm}} * \left(1 - \frac{2\sqrt{2}}{\sqrt{N_{0pm}}}\right)$$

although, especially for low values of $N_{0pm}$ , this value is much less accurate than the product $Dh_{N0pm}$ of the individual densities indicated by h).

Now to prove the asymptotic quasi-equivalence between the product $Dh_{N0pm}$ and the density $Dnc_{]0,\,N_{0pm}\,]}$ we calculate the limit for $N_{0pm} \to \infty$ of the ratio between the density $Dh_{N0pm}$ and that $Dnc_{]0,\,N_{0pm}\,]}$:

$$\lim_{N_{0pm}\to\infty} \frac{\prod_{p=2}^{pmax}\left(1-\frac{1}{p}-\frac{p-[N_{0pm}]_p}{N_{0pm}*p}\right)}{\frac{1}{\log N_{0pm}}*\left(1-\frac{2\sqrt{2}}{\sqrt{N_{0pm}}}\right)} = \frac{\lim_{N_{0pm}\to\infty}\prod_{p=2}^{pmax}\left(1-\frac{1}{p}-\frac{p-[N_{0pm}]_p}{N_{0pm}*p}\right)}{\lim_{N_{0pm}\to\infty}\frac{1}{\log N_{0pm}}*\left(1-\frac{2\sqrt{2}}{\sqrt{N_{0pm}}}\right)} = \frac{\lim_{N_{0pm}\to\infty}\prod_{p=2}^{pmax}\left(1-\frac{1}{p}-\frac{p-[N_{0pm}]_p}{N_{0pm}*p}\right)}{\lim_{N_{0pm}\to\infty}\frac{1}{\log N_{0pm}}*\left(1-\frac{2\sqrt{2}}{\sqrt{N_{0pm}}}\right)} =$$

$$= \frac{\lim_{N_{0pm}\to\infty}\prod_{p=2}^{pmax}\left(1-\frac{1}{p}\right) - \lim_{N_{0pm}\to\infty}\frac{p-[N_{0pm}]_p}{N_{0pm}*p}}{\lim_{N_{0pm}\to\infty}\frac{1}{\log N_{0pm}} * \lim_{N_{0pm}\to\infty}\left(1-\frac{2\sqrt{2}}{\sqrt{N_{0pm}}}\right)} = \frac{\lim_{N_{0pm}\to\infty}\prod_{p=2}^{pmax}\left(1-\frac{1}{p}\right) - 0}{\lim_{N_{0pm}\to\infty}\frac{1}{\log N_{0pm}}*1} = \frac{\lim_{N_{0pm}\to\infty}\prod_{p\le\sqrt{2*N_{0pm}}}\frac{(p-1)}{p}}{\lim_{N_{0pm}\to\infty}\frac{1}{2*\log\sqrt{2*N_{0pm}}-\log 2}} =$$

$$= \lim_{N_{0pm}\to\infty}\frac{\prod_{p\le\sqrt{2*N_{0pm}}}\frac{(p-1)}{p}}{\frac{1}{2*\log\sqrt{2*N_{0pm}}}} = 2 * \lim_{N_{0pm}\to\infty}\log\sqrt{2*N_{0pm}} * \prod_{p\le\sqrt{2*N_{0pm}}}\frac{(p-1)}{p} = 2*e^{-\gamma} = 1{,}122 \cong 1$$

knowing that by Merten's third theorem the limit $\lim_{N_{0pm}\to\infty}\log\sqrt{2*N_{0pm}} * \prod_{p\le\sqrt{2*N_{0pm}}}\frac{(p-1)}{p}$ is equal to $e^{-\gamma}$ with γ Euler-Mascheroni constant equal to 0.57721 ..........

For the purposes of demonstration, this relationship, although weaker than asymptotic equivalence, allows us to pose $Dh_{N0pm} \approx Dnc_{]0,\,N_{0pm}\,]}$.

**Lemma (c)** *The product $Di_{N0pm}$ and the square of $Dh_{N0pm}$ are asymptotically almost equivalent functions (with a relative error of 2*$C_2$ ).*

To prove the asymptotic quasi-equivalence between the product $Di_{N0pm}$ and the square of the density $Dh_{N0pm}$ we calculate the limit for $N_{0pm} \to \infty$ of the ratio between the density $Di_{N0pm}$ and the square of the density $Dh_{N0pm}$:

$$\lim_{N_{0pm}\to\infty}\frac{\frac{1}{2}\prod_{p=3}^{pmax}\left(1-\frac{2}{p}-\frac{2*(p-[N_{0pm}]_p)}{N_{0pm}*p}\right)}{\prod_{p=2}^{pmax}\left(1-\frac{1}{p}-\frac{p-[N_{0pm}]_p}{N_{0pm}*p}\right)^2} = \frac{\lim_{N_{0pm}\to\infty}\frac{1}{2}\prod_{p=3}^{pmax}\left(1-\frac{2}{p}-\frac{2*(p-[N_{0pm}]_p)}{N_{0pm}*p}\right)}{\lim_{N_{0pm}\to\infty}\prod_{p=2}^{pmax}\left(1-\frac{1}{p}-\frac{p-[N_{0pm}]_p}{N_{0pm}*p}\right)*\left(1-\frac{1}{p}-\frac{p-[N_{0pm}]_p}{N_{0pm}*p}\right)} =$$

$$\frac{\lim_{N_{0pm}\to\infty}\frac{1}{2}\prod_{p=3}^{pmax}\left(1-\frac{2}{p}\right) - \lim_{N_{0pm}\to\infty}\frac{1}{2}\prod_{p=3}^{pmax}\left(\frac{2*(p-[N_{0pm}]_p)}{N_{0pm}*p}\right)}{\lim_{N_{0pm}\to\infty}\prod_{p=2}^{pmax}\left(1-\frac{1}{p}-\frac{p-[N_{0pm}]_p}{N_{0pm}*p}\right)*\lim_{N_{0pm}\to\infty}\prod_{p=2}^{pmax}\left(1-\frac{1}{p}-\frac{p-[N_{0pm}]_p}{N_{0pm}*p}\right)} =$$

$$\frac{\lim_{N_{0pm}\to\infty}\frac{1}{2}\prod_{p=3}^{p_{max}}\left(1-\frac{2}{p}\right)}{\lim_{N_{0pm}\to\infty}\prod_{p=2}^{p_{max}}\left(1-\frac{1}{p}\right)^2}=\frac{\lim_{N_{0pm}\to\infty}\frac{1}{2}\prod_{p=3}^{p_{max}}\left(\frac{p-2}{p}\right)}{\lim_{N_{0pm}\to\infty}(\prod_{p=2}^{p_{max}}\frac{p-1}{p})^2}=\frac{\lim_{N_{0pm}\to\infty}\frac{1}{2}\prod_{p=3}^{p_{max}}\left(\frac{p-2}{p}\right)}{\lim_{N_{0pm}\to\infty}\frac{1}{4}\prod_{p=3}^{p_{max}}(\frac{p-1}{p})^2}=\lim_{N_{0pm}\to\infty}\frac{1}{2}\prod_{p=3}^{p_{max}}\left(\frac{p-2}{p}\right)4*$$

$$(\frac{p}{p-1})^2=2*\lim_{N_{0pm}\to\infty}\prod_{p=3}^{p_{max}}\left(\frac{p(p-2)}{(p-1)^2}\right)\approx 2*C_2\approx 1{,}32$$

For the purposes of demonstration, this relationship, although weaker than the asymptotic equivalence, allows us to pose from here on $Di_{N0pm} \approx Dh_{N0pm}{}^2$.

**Lemma (d) We prove that $Di_{N0pm}$ e $Dncncomp_{]0,\ N_{0pm}]}$ *are asymptotically equivalent.***

Being:

1) $\lim_{N_{0pm}\to\infty}\frac{Dh_{N0pm}}{Dnc_{]0,\,N_{0pm}\,]}} \approx 2*e^{-\gamma}$ (Lemma b)

2) $\lim_{N_{0pm}\to\infty}\frac{Di_{N0pm}}{(Dh_{N0pm})^2}\approx 2*C_2$ (Lemma c)

based on points 1) and 2) we can write that:

3) $\lim_{N_{0pm}\to\infty}\frac{Di_{N0pm}}{\left(Dnc_{]0,\,N_{0pm}\,]}\right)^2}=\lim_{N_{0pm}\to\infty}\frac{Di_{N0pm}}{(Dh_{N0pm})^2}*\frac{(Dh_{N0pm})^2}{\left(Dnc_{]0,\,N_{0pm}\,]}\right)^2}=\lim_{N_{0pm}\to\infty}\frac{Di_{N0pm}}{\left(Dnc_{]0,\,N_{0pm}\,]}\right)^2}\approx$

$\approx 2*C_2*(2e^{-\gamma})^2 \approx 1{,}66$

and thanks to (1) and (u) we can write:

4) $\frac{Di_{N0pm}}{Dnc_{]0,\,N_{0pm}\,]}{}^2}\approx\frac{Dncncomp_{]0,\sqrt{2N_{0pm}}\#]}}{(Dnc_{]0,\sqrt{2N_{0pm}}\#]})2}$

namely that the two ratios $\frac{Di_{N0pm}}{Dnc_{]0,\,N_{0pm}\,]}{}^2}$ e $\frac{Dncncomp_{]0,\sqrt{2N_{0pm}}\#]}}{(Dnc_{]0,\sqrt{2N_{0pm}}\#]})2}$ are asymptotically almost equivalent.

Ultimately then based on the fact that:

- $Di_{N0pm}$ approximates without equalling the density $Dncncomp_{]0,\,N_{0pm}\,]}$ (Lemma a)
- $Di_{N0pm}$ e $\left(Dnc_{]0,\,N_{0pm}\,]}\right)^2$ are asymptotically almost equivalent (formula 3)
- the two reports $\frac{Di_{N0pm}}{Dnc_{]0,\,N_{0pm}\,]}{}^2}$ e $\frac{Dncncomp_{]0,\sqrt{2N_{0pm}}\#]}}{(Dnc_{]0,\sqrt{2N_{0pm}}\#]})2}$ are asymptotically almost equivalent

we could deduce that $Di_{N0pm} \approx Dncncomp_{]0,\ N_{0pm}]}$ but this remains an open point.

Ultimately [see w) and x)] the ratio $\frac{Di_{N0pm}}{Dh_{N0pm}{}^2}\approx\frac{Dncncomp_{]0,\,N_{0pm}\,]}}{Dnc_{]0,\,N_{0pm}\,]}{}^2}$ is always greater than 1 and therefore (4.2.8) and (4.2.19) of the text are proved by substituting the value $2*C_2$ for that $2*C_2*(1-a_{rt})$.

*APPENDIX C*

Wanting to give a numerical example, let us set the first M = 41 and for ease of exposition, let us indicate the interval ]0, M] = ]0, 41] in the manner shown below where they are tiled:

- right the numbers $n_0$ prisotto of M=41
- on the left the numbers $n_0$ of which 1 is incongruous (nc1) and incompcongruous (ncomp1)

placed in the row respectively with the numbers in the range ]0, 41] having those characteristics

| n0 of which 1 is incompcongruous(ncomp1) and incongruous (nc1) | | | M | | n0 prisotto of M(NC) |
|---|---|---|---|---|---|
| 41 | | | 41 | NC | 0 |
| 40 | ncomp1 | | 40 | | 1 |
| 39 | | | 39 | | 2 |
| 38 | | nc1 | 38 | | 3 |
| 37 | | | 37 | NC | 4 |
| 36 | ncomp1 | | 36 | | 5 |
| 35 | | | 35 | | 6 |
| 34 | | | 34 | | 7 |
| 33 | | | 33 | | 8 |
| 32 | | nc1 | 32 | | 9 |
| 31 | | | 31 | NC | 10 |
| 30 | ncomp1 | nc1 | 30 | | 11 |
| 29 | | | 29 | NC | 12 |
| 28 | ncomp1 | | 28 | | 13 |
| 27 | | | 27 | | 14 |
| 26 | | | 26 | | 15 |
| 25 | | | 25 | | 16 |
| 24 | | nc1 | 24 | | 17 |
| 23 | | | 23 | NC | 18 |
| 22 | ncomp1 | | 22 | | 19 |
| 21 | | | 21 | | 20 |
| 20 | | nc1 | 20 | | 21 |
| 19 | | | 19 | NC | 22 |
| 18 | ncomp1 | nc1 | 18 | | 23 |
| 17 | | | 17 | NC | 24 |
| 16 | ncomp1 | | 16 | | 25 |
| 15 | | | 15 | | 26 |
| 14 | | nc1 | 14 | | 27 |
| 13 | | | 13 | NC | 28 |
| 12 | ncomp1 | nc1 | 12 | | 29 |
| 11 | | | 11 | NC | 30 |
| 10 | ncomp1 | | 10 | | 31 |
| 9 | | | 9 | | 32 |
| 8 | | nc1 | 8 | | 33 |
| 7 | | | 7 | NC | 34 |
| 6 | ncomp1 | nc1 | 6 | | 35 |

| | | | | |
|---|---|---|---|---|
| 5 | first | 5 | first | 36 |
| | minors of | | minors | |
| 4 | | 4 | of | 37 |
| 3 | Rad(M) | 3 | Rad(M) | 38 |
| 2 | | 2 | | 39 |
| 1 | | 1 | | 40 |
| 0 | | 0 | | 41 |

In the table there are 10 $n_0$ prisotto of M (NC), 10 $n_0$ of which 1 is not congruous (nc1), 10 $n_0$ of which 1 is not compcongruous (ncomp1) and 3 $n_0$ of which 1 is not congruous and is not compcongruous (PG) for which $n_0$ +1 and $n_0$ -1 are primes.
It is easily verified from the table that the verified density $(Dnc_{]0,M]ver})^2$=10/41=0.2439; Dncncomp(PG)(ver) =3/41=0.0731; PG(ver)= 3.

On the other hand, it follows from the NPT and (3.3.8) that:

- the calculated density $Dnc_{]0,M]calc}$ = (M/lnM - Rad(M)/lnRad(M))/M = [1 - (2/Rad(M)]*(1/lnM) = [1-(2/Rad(41)]*1/ln41=0.1851 (to the nearest 24.10 %)
- Dncncomp(PG)(calc) =1.32*.$(Dnc_{]0,M]calc})^2$ = 0.0452 (to the nearest 38.16 %)
- PG(calc) = 41* Dncncomp(PG)(calc) = 1 (to the nearest 66 %)

But with M ≥ 53 (first prime greater than 49), the values of the above quantities change along with their approximations as shown in the example table:

| **M** | **$Dnc_{]0,M]ver}$** | **$Dnc_{]0,M]calc}$** | **Appr. %** | **Dncncomp(PG) (ver)** | **Dncncomp(PG) (calc)** | **Appr. %** | **PG (ver)** | **PG (calc)** | **Appr. %** |
|---|---|---|---|---|---|---|---|---|---|
| 53 | 0,226 | 0,182 | 19,32 | 0.0754 | 0,0440 | 41,63 | 4 | 2 | 50,0 |
| 20047 | 0,111 | 0,995 | 10,65 | 0,0163 | 0,0130 | 20,09 | 328 | 262 | 20,0 |
| 40009 | 0,103 | 0,093 | 10,11 | 0,0143 | 0,0115 | 19,56 | 573 | 460 | 19,72 |

***APPENDIX D***

Wanting to give a numerical example, let us assume the first M = 41 and for ease of exposition, let us indicate the interval ]0, 2M] = ]0, 82] in the manner shown below where they are tiled:

- Odd numbers between 0 and 82 as, apart from 2, there are only prime numbers among them
- The numbers $n_0$ prisotto (nc) and prisopra (ncomp) of the interval ]0, 41] placed in a row with the prime numbers of the interval ]0, 41] and with the prime numbers of the interval [41, 82], respectively.

| n0 prisopra (ncom) and prisotto (nc) | | | | M | |
|---|---|---|---|---|---|
| 0 | ncomp | nc | | 41 | |
| 2 | ncomp | | 39 | | 43 |
| 4 | | nc | 37 | | 45 |
| 6 | ncomp | | 35 | | 47 |
| 8 | | | 33 | | 49 |
| 10 | | nc | 31 | | 51 |
| 12 | ncomp | nc | 29 | | 53 |
| 14 | | | 27 | | 55 |
| 16 | | | 25 | | 57 |
| 18 | ncomp | nc | 23 | | 59 |
| 20 | ncomp | | 21 | | 61 |
| 22 | | nc | 19 | | 63 |
| 24 | | nc | 17 | | 65 |
| 26 | ncomp | | 15 | | 67 |
| 28 | | nc | 13 | | 69 |
| 30 | ncomp | nc | 11 | | 71 |
| 32 | ncomp | | 9 | | 73 |
| 34 | | first | 7 | | 75 |
| | | minors | | | |
| 36 | | of | 5 | | 77 |
| | | | | | |
| 38 | ncomp | Rad(2M) | 3 | | 79 |
| | | | 1 | | 81 |

In the table, there are 9 $n_0$ prisotto (nc) of M, 10 $n_0$ prisopra (ncomp) of M and 4 $n_0$ prisotto and prisopra ($M_G$ ) of M for which 41+ n and$_0$ 41- $n_0$ are primes.
It is easily verified from the table that the verified density $Dnc_{]0,M]ver}$ = 9/41=0.2195; Dncncomp($M_G$ )(ver) =4/41=0.0975; $M_G$ (ver) = 4.

On the other hand, it follows from the NPT and (4.2.8) that:

- The calculated density $Dnc_{]0,M]calc}$= (M/lnM - Rad(2M)/lnRad(2M))/M = [1 - (2*Rad(2)/Rad(M)]*(1/lnM) = [1-(2*Rad(2)/Rad(41)]*1/ln41=0.1502 (to the nearest 31.57 %)

- Dncncomp($M_G$ )(calc) =1.32*$(Dnc_{]0,M]calc})^2$ = 0.0297 (to the nearest 69.53 %)

- $M_G$ (calc) = 41* Dncncomp($M_G$ )(calc) = 1 (to the nearest 75 %)

But with M ≥ 127 (first prime greater than 121), the values of the above quantities change along with their approximations as shown in the example table:

| **M** | **Dnc**$_{]0,M]ver}$ | **Dnc**$_{]0,M]calc}$ | **Appr. %** | **Dncncomp(**M )$_G$ **(ver)** | **Dncncomp(**M )$_G$ **(calc)** | **Appr %** | **M$_G$ (ver)** | **M$_G$ (calc)** | **Appr %** |
|---|---|---|---|---|---|---|---|---|---|
| 127 | 0,196 | 0,1546 | 21,45 | 0,0629 | 0,0315 | 49,9 | 8 | 4 | 50,0 |
| 20047 | 0,110 | 0,0989 | 10,70 | 0,0146 | 0,0129 | 11,6 | 293 | 259 | 11,6 |
| 40009 | 0,103 | 0,0930 | 10,18 | 0,0127 | 0,0114 | 10,3 | 510 | 457 | 10,4 |

*APPENDIX E*

We know that according to combinatorial calculus $(Dnc_{]0,\sqrt{2N_{0pm}}\#]})$ e $(Dncncomp_{]0,\sqrt{2N_{0pm}}\#]})$ are respectively equal to:

$$\prod_{p=2}^{p_{max}} \frac{(p-1)}{p} \qquad \text{e } \frac{1}{2} * \prod_{p=3}^{pmax} \frac{(p-2)}{p}$$

with pmax the first higher less than $\sqrt{2N_{0pm}}$

By developing the $Dncncomp_{]0,\sqrt{2N_{0pm}}\#]}$ (but would be entirely analogous by developing the $Dnc_{]0,\sqrt{2N_{0pm}}\#]}$) and multiplying and dividing all the terms of the product by $p_{max}\#$we obtain:

$$Dncncomp_{]0,\sqrt{2N_{0pm}}\#]} = \frac{1}{2} * \prod_{p=3}^{pmax} \frac{(p-2)}{p} = \frac{1}{2} * \frac{1}{3} * \frac{3}{5} * \frac{5}{7} * \dots \dots . . * \frac{p_{max}-2}{p_{max}} =$$

$$\frac{p_{max}\#/2}{p_{max}\#} * \frac{p_{max}\#/3}{p_{max}\#} * \frac{3 * p_{max}\#/5}{p_{max}\#} * \frac{5 * p_{max}\#/7}{p_{max}\#} * \dots \dots \dots . . * \frac{(p_{max}-2) * p_{max}\#/p_{max}}{p_{max}\#}$$

where the individual terms of the final product represent the individual densities in the range ]0, $p_{max}\#$] of the incongruous and incompcongruous numbers with $N_{0pm}$ mod. p_1. Similarly, if we refer to the density $\text{Dncncomp}_{]0,\text{N}_{\text{opm}}]}$ it will be roughly given by the product of the individual densities D2 (p_1). This small approximation is due to the fact that both the single density (p-2)/p of the incongruous and incompcongruous with Nopm modulus p and the total density of the incongruous and incompcongruous with Nopm ∏(p-2)/p are correct for combinatorial calculations as long as the amplitude of the interval to which they refer is respectively a multiple of the single p or of all p ≤ pmax. In our interval ]0, Nopm], on the other hand, the amplitude is not a multiple of any p ≤ pmax and this results in both the small difference between Dncncomp]0, Nopm] and Dncncomp]0, pmax #] and the difference between Dinopm and the density Dncncomp]0, Nopm].
For the sake of clarity, let us give a tabular example based on the development of the various possible combinations of the incongruous numbers with $N_{0pm}$ modulo p_1 for all p_1 ≤ $p_{max}$. In the example we refer for simplicity to the calculation of Dnc]0, Nopm] (for that of Dncncomp]0, Nopm] the reasoning is entirely analogous) and choose a small value of $N_{0pm}$. The example is shown in figure A.

Nopm=23; pmax<RADQ(2*23) ---> pmax=5; [23]mod2=1; [23]mod3=2; [23]mod5=3

| n | p=2 | p=3 | p=5 |
|---|---|---|---|
| 1 | 1 | 1 | 1 |
| 2 | 0 | 2 | 2 |
| 3 | 1 | 0 | 3 |
| 4 | 0 | 1 | 4 |
| 5 | 1 | 2 | 0 |
| 6 | 0 | 0 | 1 |
| 7 | 1 | 1 | 2 |
| 8 | 0 | 2 | 3 |
| 9 | 1 | 0 | 4 |
| 10 | 0 | 1 | 0 |
| 11 | 1 | 2 | 1 |
| 12 | 0 | 0 | 2 |
| 13 | 1 | 1 | 3 |
| 14 | 0 | 2 | 4 |
| 15 | 1 | 0 | 0 |
| 16 | 0 | 1 | 1 |
| 17 | 1 | 2 | 2 |
| 18 | 0 | 0 | 3 |
| 19 | 1 | 1 | 4 |
| 20 | 0 | 2 | 0 |
| 21 | 1 | 0 | 1 |
| 22 | 0 | 1 | 2 |
| 23 | 1 | 2 | 3 |
| 24 | 0 | 0 | 4 |
| 25 | 1 | 1 | 0 |
| 26 | 0 | 2 | 1 |
| 27 | 1 | 0 | 2 |
| 28 | 0 | 1 | 3 |
| 29 | 1 | 2 | 4 |
| 30 | 0 | 0 | 0 |

figura A-4

| n | p=2 |
|---|---|
| 1 | 1 |
| 2 | 0 |

figura A-1

| n | p=3 |
|---|---|
| 1 | 1 |
| 2 | 2 |
| 3 | 0 |

figura A-2

| n | p=5 |
|---|---|
| 1 | 1 |
| 2 | 2 |
| 3 | 3 |
| 4 | 4 |
| 5 | 0 |

figura A-3

| n | 2 | 3 |
|---|---|---|
| 1 | 1 | 1 |
| 2 | 0 | 2 |
| 3 | 1 | 0 |
| 4 | 0 | 1 |
| 5 | 1 | 2 |
| 6 | 0 | 0 |

figura A-5

Figures A-1, A-2 and A-3 show with coloured backgrounds the n incongruous with Nopm for modules 2, 3 and 5 respectively in the intervals [1, 2], [1, 3] and [1, 5]. Wanting to calculate the incongruous numbers with Nopm for modules 2, 3 and 5, we first see in Fig. A-5 how the incongruous numbers with Nopm for module 2 combine with the incongruous numbers for module 3 in the interval [1, 2*3]. We observe how, due to the primality of 2 and 3, the incongruous 0 of modulus 2 combines in the interval [1, 6] with all values of modulus 3 and thus also with the incongruous 1 and 0 of modulus 3. In the interval [1, 6] the total incongruity combinations (for modulo 2 and modulo 3) are 2 = (2-1)*(3-1) and the Dnc density of incongruous numbers with Nopm in the same interval is 2/6 = 1/3

Thus we see in Figure A-4 how the two incongruities mod. 2 and mod. 3 of the interval [1, 6] combine in the interval [1, 2*3*5], where 2*3*5=30 = pmax #, with all values of modulus 5 and thus also with the incongruities 0, 1, 2 and 4 of modulus 5. In the interval ]0, 30] the total incongruity combinations (for modulo 2, modulo 3 and modulo 5) are 8 = (2-1)*(3-1)*(5-1) and the density Dnc]0, pmax #] of incongruous numbers with Nopm in the same interval is 8/30 = 4/15 = 0.2666....

Let us now consider, instead of the interval ]0, 30] = ]0, pmax #], the interval ]0, 23] = ]0, Nopm] and compare the density Dnc]0, Nopm] of incongruities with Nopm relative to the afore mentioned interval and obtainable from Fig. A-4 with that Dhnopm calculable as the product of the individual densities D(p_1) (see Appendix B (a)) relative to modules 2, 3, 5.

From Figure A-4 we derive that the incongruous numbers with Nopm (23) in the interval ]0, 23] are 6 and that therefore the density Dnc]0, Nopm] is 6/23 = 0.2608....

To calculate Dhnopm instead, we calculate the individual densities D(p_1) of the incongruous numbers with Nopm for modules 2, 3 and 5 respectively:

D(p_1) = (L*(p_1 - 1) + [Nopm]p_1 - h)/Nopm

where L is equal to the ratio between the maximum multiple Xp_1 of p_1 contained in the interval ]0, N0pm] and p_1; where [N0pm]p_1 is the class of Nopm modulo p_1 and is equal to the amplitude of the interval ]Xp_1, Nopm]; where h is the number of congruent numbers present in the interval ]Xp_1, Nopm] and which in our case is equal to 1, Nopm being a number congruent with itself. We will therefore have:

D(2) = (11*1+(1-1))/23 = 11/23; D(3) = (7*2+(2-1))/23 = 15/23; D(5) = (4*4+(3-1))/23 = 18/23

and thus: Dhnopm = D(2)*D(3)*D(5) = 11/23 * 15/23 * 18/23 = 0.2441

From this comparison, we deduce that the relative approximation between Dnc]0, Nopm] and Dhnopm is approximately 6.4 %, and it can easily be verified that the above approximation holds for increasing values of Nopm well below 10 %.

This approximation is due to the fact that both the single density (p-1)/p of the incongruities with Nopm modulo p and the total density of the incongruities with Nopm ∏(p-1)/p are correct for combinatorial calculations as long as the amplitude of the interval to which they refer is respectively a multiple of the single p or of all p ≤ pmax. In our interval ]0, Nopm], on the other hand, the amplitude is not a multiple of any p ≤ pmax and this results in both the small difference between Dnc]0, Nopm] and Dnc]0, pmax #] and that between Dhnopm and the density Dnc]0, Nopm] of the incongruous numbers with Nopm moduli 2, 3 and 5

The same reasoning with similar conclusions can be made about the approximation between Dncncomp]0, Nopm] and Dinopm

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